Improved interface condition for $2D$ Domain Decomposition with corner: a theoretical determination

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Abstract: This article deals with a local improvement of domain decomposition methods for 2-dimensional elliptic problems for which either the geometry or the domain decomposition presents conical singularities. The problems amounts to determining the coefficients of some interface boundary conditions so that the domain decomposition algorithm converges rapidly. Specific problems occur in the presence of conical singularities. Starting from the method used for regular interfaces, we derive a local improvement by matching the singularities, that is the first terms of the asymptotic expansion around the corner, provided by Kondratiev theory. This theoretical approach leads to the explicit computation of some coefficients in the interface boundary conditions, to be tested numerically. This final numerical step is presented in a companion article. This part focuses on the method used to compute these coefficients and provides detailed examples on a model problem.

1 Introduction.

Domain decomposition methods constitute a very active research field in numerical analysis. They are now well understood in the case of a regular domain decomposed into regular subdomains, see for example [2], [10], [14], [19], [21] and [22]. A significant challenge for the applications is a good understanding of the singular cases, for example problems with corners in 2D. The general principle of those methods is as follows:

1. For an elliptic differential operator L, a domain Ω and a given right hand side f, consider the problem of finding u such that
$$\left\{ \begin{array}{ll} Lu & = f \text{ in } \Omega \\ & +B.C. \text{ on } \partial\Omega\,. \end{array} \right. \tag{1}$$
2. When the domain Ω is large from the numerical complexity point of view or complex from the modeling or geometrical point of view, it can be decomposed into subdomains, $\overline{\Omega} = \cup_{i=1}^{i=N} \overline{\Omega_i}$ where each Ω_i is an open subdomain of Ω.
3. The initial problem (1) is then approximated by an iterative process: the step $n+1$ is determined by solving
$$\left\{ \begin{array}{rlcl} Lu_i^{n+1} & = f & \text{in} & \Omega_i \\ B_{ij}\gamma_{ij}u_i^{n+1} & = B_{ij}\gamma_{ij}u_j^n & \text{on} & \partial\Omega_i \cap \overline{\Omega_j} (i \neq j) \\ & +B.C. & \text{on } \partial\Omega_i \cap \partial\Omega & \end{array} \right. \tag{2}$$
simultaneously in all subdomains Ω_i, $i = 1, \ldots, N$.

The interface operators B_{ij} can be differential or pseudodifferential matricial operators applied to some trace vector $\gamma_{ij}u$. The choice of the interface operators has a very great influence on the speed of convergence of the algorithm. Cases are known where for a given geometry the change of B_{ij} transforms the absence of convergence into the exact convergence after a finite number of iterations. Within the framework of the regular interfaces, this analysis has already been done theoretically and numerically. A good final choice actually relies on a compromise between the theoretical optimality and the ease of implementation, see [12][19].

In a domain with a conical singularity (corner), it is known after Kondratiev [13] that even when the left-hand side of (1) vanishes at infinite order at the corner the solution may have singularities or more generally a non trivial asymptotic expansion. Moreover the first term of this asymptotic expansion at the corner depends strongly on the operator L, the geometry which is reduced to the corner angle as a first approximation, and the boundary conditions. A priori the singularities of the solutions to the elliptic boundary value problem in the subdomains Ω_i, $i = 1 \dots N$, do not coincide with the ones corresponding to the problem in the whole domain Ω. The diagnosis of a locally slower convergence of domain decomposition algorithm in the presence of conical singularities in [21] invoked this bad matching. Again the first terms of the asymptotic expansion around the corner of a solution to an elliptic boundary problem depends on the boundary conditions, namely the operators B_{ij} occurring in the subdomain problems. The present work again makes use of the flexibility in the choice of the interface boundary conditions B_{ij}, on which the singularities depend, in order to improve the convergence in such cases.

The leading idea is that the iterative process must not produce artificial singularities at the step $n+1$ if the approximate solutions $(u_i^n)_{i=1\dots N}$ at step n have the right asymptotic expansion corresponding to the whole domain Ω.

A short review of Kondratiev theory for elliptic boundary value problems in domain with conical singularities (see for example [5],[9], [13]), provides the accurate meaning to "singularities", "asymptotic expansion" or "asymptotic type". The matching of the first terms of the asymptotic expansion around the corner between the whole domain problem and the subdomain problems leads to some relations which have to be fulfilled by the coefficients of the interface boundary conditions $B_{ij} \circ \gamma_{ij}$. This article focuses on the computation of these relations and presents the general strategy for the improvement of the efficiency of domain decomposition methods around corners. The numerical tests which validate definitely this approach will be given in a forthcoming article.

For the sake of simplicity, our analysis will be restricted to the model operator $\eta - \Delta$ in a sectorial domain $\mathbb{R}_+^* \times (\theta_0, \theta_+) = \Omega \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$. A more general polygonal domain would not change the analysis which is local around every corner. Every new case requires some specific calculations which can be far from explicit in a general framework. We preferred to make the strategy more obvious by treating as accurately as possible those specific examples. The method can be extended to other cases and some other problems motivated by numerical applications are deferred to another article.

The article is organized as follows: After the presentation of the asymptotic point of view which support the whole analysis (Section 2), we check that the variational formulation of subdomain problems makes sense for some quite general second order interface boundary conditions (Section 3). Then we review the main theoretical tools related with Kondratiev theory (Section 4 and 5). The Section 6 presents roughly the general strategy and provides some definition. The rest of the article (Section 7) is devoted to the full treatment of some specific domain decomposition problems with this approach.

2 The problem.

We work in the sectorial domain $\Omega \subset \mathbb{R}^2$ written in polar coordinates $\Omega = \mathbb{R}_+^* \times (\theta_0, \theta_+)$, or $\Omega = \mathbb{R}^2 = \mathbb{R}_+^* \times S^1 \cup \{O\}$ where the origin O corresponds to $r = 0$. On this simple geometry, we consider the model boundary value problem (1) with $L = \eta - \Delta$, $\eta > 0$. For such order 2 elliptic partial differential operators and when the interfaces are regular, the practically used and efficient interface boundary conditions involve the second order tangential derivative according to

$$B_{ij}(\frac{\partial u}{\partial \nu}, u) = \frac{\partial u}{\partial \nu} + \beta u - \frac{\partial}{\partial \tau}(\frac{\alpha}{2}\frac{\partial}{\partial \tau})u$$

where $\frac{\partial}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$, are the normal derivative and the tangential derivative respectively.

When $\Omega = \mathbb{R}^2$ is split into two half-planes and for a given bounded set of frequencies in the tangential variable, proportional to $1/h$ when h is the characteristic mesh size of the numerical discretization, there is an optimal choice of (α, β) for the convergence of the domain decomposition process (2). By introducing the error at step n, $e_i^n(x, \tau) = u_i^n - u$ and its Fourier transform in the tangential variable $\widehat{e_i^n}(x, k)$, the optimized coefficients $\alpha_{opt} > 0$ and $\beta_{opt} > 0$ are determined according to [12] by the max-min principle

$$\min_{\alpha, \beta \in \mathbb{R}} \max_{|k| \leq \frac{\pi}{h}} |\rho(k; \alpha, \beta)| \tag{3}$$

with $\rho(k; \alpha, \beta) = \frac{\widehat{e_i^{n+2}}(0,k)}{\widehat{e_i^n}(0,k)}$.

Here our domain $\Omega = \mathbb{R}_+^* \times (\theta_0, \theta_+)$ or $\Omega = \mathbb{R}^2$, is decomposed into tow sectors $\Omega_1 = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ and $\Omega_2 = \mathbb{R}_+^* \times (\theta_0, \theta_-)$ (with the convention $\theta_0 = \theta_+ - 2\pi$ when $\Omega = \mathbb{R}^2$). We shall use polar coordinates $(r, \theta) \in \mathbb{R}_+^* \times [\theta_0, \theta_+]$ in order to develop the analysis around the corner, $r = 0$, of this domain decomposition. The previous boundary operator now takes the form

$$B_{ij}(\pm \frac{\partial}{r\partial\theta} u, u) = \pm \frac{\partial}{r\partial\theta} u + \beta_{opt} u - \frac{\alpha_{opt}}{2} \frac{\partial^2}{\partial r^2} u\,. \tag{4}$$

More precisely, the boundary problem in Ω_1 with the interface condition (4) solved by the error $e_1^{n+1} = u_1^{n+1} - u$ reads

$$\begin{cases} \left(\eta - \frac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2)\right) e_1^{n+1}(r,\theta) &= 0 \\ \left(-\frac{1}{r}\partial_\theta - \beta_{opt} + \frac{1}{2}\partial_r(\alpha_{opt}\partial_r)\right) e_1^{n+1}(r,\theta_+) &= \left(-\frac{1}{r}\partial_\theta - \beta_{opt} + \frac{1}{2}\partial_r(\alpha_{opt}\partial_r)\right) e_2^{n}(r,\theta_+) \\ \left(-\frac{1}{r}\partial_\theta + \beta_{opt} - \frac{1}{2}\partial_r(\alpha_{opt}\partial_r)\right) e_1^{n+1}(r,\theta_-) &= \left(-\frac{1}{r}\partial_\theta + \beta_{opt} - \frac{1}{2}\partial_r(\alpha_{opt}\partial_r)\right) e_2^{n}(r,\theta_-) \end{cases} \tag{5}$$

One boundary condition $(B_{ij} \circ \gamma_{ij})$ is possibly replaced by the global one along $\partial\Omega \setminus \partial\Omega_1$. The same is done in Ω_2 (and all other subdomains).

2.1 Principal part and homogeneity.

The aim of the analysis is made in order to improve locally around the corner of subdomains the behavior of domain decomposition algorithms. Hence we will work with the asymptotic expansions as $r \to 0$ of the solutions to the global problem (1) and to the subdomain problems (2). It is known that the main term of such asymptotic expansions are determined by the principal part of the boundary problem determined after the homogeneity degree of the operator and boundary conditions as $r \to 0$ (see [13] and Section 5). Concerned with the homogeneity degree, the operators $r\partial_r$ and ∂_θ are operators of order 0, $\partial_r = \frac{1}{r} r\partial_r$ and $\frac{1}{r}\partial_\theta$ of order -1 and $\partial_r^2 = \frac{1}{r^2}\left[(r\partial_r)^2 - r\partial_r\right]$ has the order -2. Hence the principal part of the operator $L = (\eta - \Delta)$ is nothing but

$$-\Delta = -\frac{1}{r^2}\left((r\partial_r)^2 + \partial_\theta^2\right)$$

where the term of order 0 has been neglected.

The same has to be done for the interface boundary condition $B_{ij} \circ \gamma_{ij}$ which occur in the subdomain problems. If one starts with the subdomain problem (5) in Ω_1, the principal part is

$$\left\{ \begin{array}{rl} \Big((r\partial_r)^2 + \partial_\theta^2)\Big)e_1^{n+1}(r,\theta) & = 0 \\ (\frac{1}{2}\partial_r^2)e_1^{n+1}(r,\theta_+) & (\frac{1}{2}\partial_r^2)e_2^{n}(r,\theta_+) \\ (-\frac{1}{2}\partial_r^2)e_1^{n+1}(r,\theta_-) & = (-\frac{1}{2}\partial_r^2)e_2^{n}(r,\theta_-) \end{array} \right.$$

The absence of a normal derivative in this principal part means that asymptotically the interface conditions behave like Dirichlet interface conditions and do not transmit well the information from one subdomain to its neighboring ones. This was considered in [21] as the explanation of a slow convergence of domain decomposition algorithm around corners. One way to solve this problem is by forcing all the terms of the boundary interface operators to have the same homogeneity degree. We will take interface boundary condition which look like $\pm\frac{\partial}{r\partial\theta} + \frac{\beta_\pm}{r} - \frac{\partial}{\partial r}\frac{\alpha_\pm r}{2}\frac{\partial}{\partial r}$ around the corner $r = 0$ with $\alpha_\pm$ and $\beta_\pm$ constant. Far from the corner, the interface boundary must keep the optimal form (4). A synthesis of this is done by taking

$$B_{ij}(\pm\frac{\partial}{r\partial\theta}u, u) = \pm\frac{\partial}{r\partial\theta}u + \tilde{\beta}_\pm(r)u - \frac{\partial}{\partial r}\frac{\tilde{\alpha}_\pm(r)}{2}\frac{\partial}{\partial r}u \tag{6}$$

$$\text{with} \qquad \tilde{\alpha}_\pm(r) = \left\{ \begin{array}{ll} \alpha_\pm r & \text{if } r \leq \frac{\alpha_{opt}}{\alpha_\pm} \\ \\ \alpha_{opt} & \text{if } r \geq \frac{\alpha_{opt}}{\alpha_\pm} \end{array} \right. \qquad \tilde{\beta}_\pm(r) = \left\{ \begin{array}{ll} \frac{\beta_\pm}{r} & \text{if } r \leq \frac{\beta_\pm}{\beta_{opt}} \\ \\ \beta_{opt} & \text{if } r \geq \frac{\beta_\pm}{\beta_{opt}} \end{array} \right. \tag{7}$$

with $\alpha_\pm > 0$ and $\beta_\pm \geq 0$.

A first practical constraint for the choice of $\alpha_\pm$ and $\beta_\pm$ is that the matching point has to be far enough from $r = 0$ in order to be effective in numerical computation. This can be written with the additional condition

$$\min\left\{\frac{\alpha_{opt}}{\alpha_\pm}, \frac{\beta_\pm}{\beta_{opt}}\right\} = \Phi(h) \quad \text{or} \quad \left(\frac{\alpha_{opt}}{\alpha_\pm} \geq \Phi(h) \text{ for } \beta_\pm = 0\right) \tag{8}$$

where the numerical discretization is represented here by the average mesh diameter $h > 0$ and the form function Φ. For example, one can consider here $\Phi(h) = 3h$ for a uniform grid or $\Phi(h) = o(h)$ as $h \to 0$ when the mesh is refined around the corner $r = 0$.
In the case where $\beta_\pm = 0$, the coefficient $\alpha_\pm$ is fully determined by the subsequent analysis and the mesh size h is chosen afterwards small enough so that $\alpha_{opt} \geq \alpha_\pm\Phi(h)$.

Finally in a subdomain Ω_1, we have to consider the boundary value problem

$$\begin{cases} \Big(\eta - \frac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2)\Big) e_1^{n+1}(r,\theta) & = 0 \\ \Big(-\frac{1}{r}\partial_\theta - \tilde{\beta}_+(r) + \frac{1}{2}\partial_r(\tilde{\alpha}_+(r)\partial_r)\Big) e_1^{n+1}(r,\theta_+) & = \Big(-\frac{1}{r}\partial_\theta - \tilde{\beta}_+(r) + \frac{1}{2}\partial_r(\tilde{\alpha}_+(r)\partial_r)\Big) e_2^{n}(r,\theta_+) \\ \Big(-\frac{1}{r}\partial_\theta + \tilde{\beta}_-(r) - \frac{1}{2}\partial_r(\tilde{\alpha}_-(r)\partial_r)\Big) e_1^{n+1}(r,\theta_-) & = \Big(-\frac{1}{r}\partial_\theta + \tilde{\beta}_-(r) - \frac{1}{2}\partial_r(\tilde{\alpha}_-(r)\partial_r)\Big) e_2^{n}(r,\theta_-) \end{cases} \tag{9}$$

whose principal homogeneous part is

$$\begin{cases} \Big((r\partial_r)^2 + \partial_\theta^2\Big) e_1^{n+1}(r,\theta) & = 0 \\ \Big(-\partial_\theta - \beta_+ + \frac{\alpha_+}{2}(r\partial_r)^2\Big) e_1^{n+1}(r,\theta_+) & = \Big(-\partial_\theta - \beta_+ + \frac{\alpha_+}{2}(r\partial_r)^2\Big) e_2^{n}(r,\theta_+) \\ \Big(-\partial_\theta + \beta_- - \frac{\alpha_-}{2}(r\partial_r)^2\Big) e_1^{n+1}(r,\theta_-) & = \Big(-\partial_\theta + \beta_- - \frac{\alpha_-}{2}(r\partial_r)^2\Big) e_2^{n}(r,\theta_-)\,. \end{cases} \tag{10}$$

Like in (5), one boundary condition $(B_{ij} \circ \gamma_{ij})$ is possibly replaced by the global one along $\partial\Omega \setminus \partial\Omega_1$. The same is done in Ω_2.

3 Variational formulation.

We specify the variational space and the variational formulation of the subdomain problem (9). Let us note that two cases will be distinguished $\beta_\pm = 0$ and $\beta_\pm \neq 0$ depending on $\Omega = \mathbb{R}_+^* \times (\theta_0, \theta_-)$ or $\Omega = \mathbb{R}^2$ and the boundary conditions. Natural variational spaces are distinct in both cases. The case $\beta_\pm \neq 0$ is not permitted in interior corner $\Omega = \mathbb{R}^2$ or more generally when the general solution of the complete problem does not vanish at $r = 0$. Nevertheless considering $\beta_\pm \neq 0$ makes sense for example when one puts homogeneous Dirichlet boundary condition on the complete problem with $\Omega = \mathbb{R}_+^* \times (\theta_0, \theta_+)$ and provides an additional flexibility in the choice of the pairs $(\alpha_\pm, \beta_\pm)$. [1]

Let Ω_1 be a sector of $\mathbb{R}^2$, $\Omega_1 = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ and we note $\overline{\Omega_1}$ its closure in $\mathbb{R}^2$. The spaces of regular test functions are $\mathcal{C}_0^\infty(\overline{\Omega_1})$, which is the space of the restrictions to Ω_1 of function $\mathcal{C}^\infty$ with compact support in $\mathbb{R}^2$, and $\mathcal{C}_0^\infty(\overline{\Omega_1} \setminus \{O\})$ the space of the elements of $\mathcal{C}_0^\infty(\overline{\Omega_1})$ whose support does not meet the corner O. These two spaces contain the space $\mathcal{C}_0^\infty(\Omega_1)$

[1]Practically, the case $\beta_\pm = 0$ is implemented by keeping a constant coefficient $\tilde{\beta}(r) = \beta_{opt}$ along the whole interface.

of interior test functions and permit the analysis of weak formulations up to the boundary (except at the corner).

The variational spaces in order to estimate all the terms involved in the variational formulation of (9) are defined according to:

- G^1 is the completion of $\mathcal{C}_0^\infty(\overline{\Omega_1})$ for the norm $\| \ \|_{G^1}$ given by

$$\|u\|_{G^1}^2 = \int_{\Omega_1} |u|^2(x) + |\nabla u|^2(x)\ dx + \int_{\partial\Omega_1} |(r\partial_r)u|^2(r)\ \frac{dr}{r}\ .$$

  This norm makes G^1 a Hilbert space in which $\mathcal{C}_0^\infty(\overline{\Omega_1})$ is dense.

- G_0^1 is the completion of $\mathcal{C}_0^\infty(\overline{\Omega_1}\setminus\{O\})$ for the norm $\| \ \|_{G_0^1}$ given by

$$\|u\|_{G_0^1}^2 = \int_{\Omega_1} |u|^2(x) + |\nabla u|^2(x)\ dx + \int_{\partial\Omega_1} |(r\partial_r)u|^2(r) + |u|^2\ \frac{dr}{r}\ .$$

  It makes G_0^1 a Hilbert space in which $\mathcal{C}_0^\infty(\overline{\Omega_1}\setminus\{O\})$ is dense.

The definition of these spaces lead moreover to the following properties:

- $G_0^1 \subset G^1 \subset H^1(\Omega_1)$.

- Hence, any $u \in G^1$ (or $u \in G_0^1$) admits a trace in $H^{\frac{1}{2}}_{loc}(\partial\Omega_1\setminus\{O\})$. The boundary terms contained in the norm of G^1 and G_0^1 specify some additional regularity and the behaviour in the vicinity of the corner for these traces.

- The inclusion $G_0^1 \subset G^1$ is strict. In fact G_0^1 does not even contain $\mathcal{C}_0^\infty(\overline{\Omega_1})$ since $\int_{\partial\Omega_1} |u(r)|^2 \frac{dr}{r} = +\infty$ for a smooth function which does not vanish in $r = 0$.

On these spaces we consider the following weak formulations where the right hand-sides f, $g_\pm$ define a linear continuous form on G^1 (resp. G_0^1):

$\beta_\pm = \mathbf{0}$:

$$\forall v \in G^1, \quad a_{\alpha_\pm,0}(u,v) = L_{f,g_-,g_+}(v) \tag{11}$$

with $a_{\alpha_\pm,0}(u,v) = \displaystyle\int_{\Omega_1} \eta u(x)v(x) + \nabla u(x)\nabla v(x)\ dx +$

$$\int_0^{+\infty} \frac{\alpha_+}{2}(r\partial_r)u(r,\theta_+)(r\partial_r)v(r,\theta_+)\ \frac{dr}{r} + \int_0^{+\infty} \frac{\alpha_-}{2}(r\partial_r)u(r,\theta_-)(r\partial_r)v(r,\theta_-)\ \frac{dr}{r}$$

and $L_{f,g_-,g_+}(v) = \displaystyle\int_{\Omega_1} f(x)v(x)\ dx - \int_0^{+\infty} g_+(r)v(r,\theta_+)\ dr + \int_0^{+\infty} g_-(r)v(r,\theta_-)\ dr\,.$

$\beta_\pm \neq \mathbf{0}$:

$$\forall v \in G^1_0, \quad a_{\alpha_\pm,\beta_\pm}(u,v) = L_{f,g_-,g_+}(v) \tag{12}$$

$$\begin{aligned}
\text{with} \quad a_{\alpha_\pm,\beta_\pm}(u,v) &= \int_{\Omega_1} \eta u(x)v(x) + \nabla u(x)\nabla v(x)\, dx + \\
&\int_0^{+\infty} \frac{\alpha_+}{2}(r\partial_r)u(r,\theta_+)(r\partial_r)v(r,\theta_+)\,\frac{dr}{r} + \int_0^{+\infty} \frac{\beta_+}{r}u(r,\theta_+)v(r,\theta_+)\, dr \\
&+ \int_0^{+\infty} \frac{\alpha_-}{2}(r\partial_r)u(r,\theta_-)(r\partial_r)v(r,\theta_-)\,\frac{dr}{r} + \int_0^{+\infty} \frac{\beta_-}{r}u(r,\theta_-)v(r,\theta_-)\, dr \\
\text{and} \quad L_{f,g_-,g_+}(v) &= \int_{\Omega_1} f(x)v(x)\, dx - \int_0^{+\infty} g_+(r)v(r,\theta_+)\, dr + \int_0^{+\infty} g_-(r)v(r,\theta_-)\, dr\,.
\end{aligned}$$

Lax-Milgram theorem provides the existence and the uniqueness of a solution.

Proposition 3.1. *For $\alpha_\pm > 0$ and $\beta_\pm = 0$ (resp. $\alpha_\pm > 0$ and $\beta_\pm > 0$) and for data (f, g_-, g_+) such as L_{f,g_-,g_+} define a linear continuous form on G^1 (resp. G^1_0) the problem (11) (resp. (12)) admits a unique solution in G^1 (resp. G^1_0). It is a weak solution to the boundary value problem*

$$\begin{cases} (\eta - \Delta)u = f \\[2mm] \left(-\frac{1}{r}\partial_\theta - \frac{\beta_+}{r} + \frac{1}{2}\partial_r(\alpha_+ r\partial_r)\right) u(r,\theta_+) = g_+(r) \\[2mm] \left(-\frac{1}{r}\partial_\theta + \frac{\beta_-}{r} - \frac{1}{2}\partial_r(\alpha_- r\partial_r)\right) u(r,\theta_-) = g_-(r)\,. \end{cases} \tag{13}$$

What occurs in the neighborhood of the corner is contained in Kondratiev theory which is summarized in the next two sections.

4 The Mellin transform and weighted Sobolev spaces.

In this section, the Mellin transform and the weighted Sobolev spaces are introduced. We point out the links with the usual Sobolev spaces. Here $\Omega = \mathbb{R}^*_+ \times \omega$ will denote a sectorial subset of $\mathbb{R}^n$, $n \geq 1$, where ω is a regular open set of S^{n-1}.

4.1 The Mellin transform.

The Mellin transform is nothing but the Fourier transformation associated with the multiplicative group $(\mathbb{R}^\star_+, .)$. The change of variable $r = e^{-t}, t \in \mathbb{R}$ makes the connection with the usual Fourier transform. It is normalized according to :

$$M(u)(\tau) = \int_0^\infty r^{i\tau} u(r) \frac{dr}{r}$$

Proposition 4.1. *1. The Mellin transform is an isometry from $L^2(\mathbb{R}_+, \frac{dr}{r})$ onto $L^2(\mathbb{R}, \frac{d\tau}{2\pi})$.*

2. Translation in the τ-variable: $M(r^{i\alpha} f)(\tau) = M(f)(\tau+\alpha)$ for $\alpha \in \mathbb{R}$.

3. Mellin transform and derivative:

$$M(ir\,\partial_r f)(\tau) = \tau M(f)(\tau), \quad M(i\ln(r) f)(\tau) = \partial_\tau M(f)(\tau). \tag{14}$$

4. Mellin transform and dilations:

$$M(f(r^c))(\tau) = c^{-1} M(f)(c^{-1}\tau), \quad M(f(c\,r)(\tau) = c^{-i\tau} M(f)(\tau), c \in \mathbb{R}_+^\star. \tag{15}$$

When f is compactly supported in $\mathbb{R}_+^*$ the Mellin transform makes sense for any $\tau \in \mathbb{C}$ and the second formula can be extended to any $\alpha \in \mathbb{C}$.

Definition 4.2. *By setting $\mathbb{C}_- = \{z \in \mathbb{C}, \mathcal{I}\mathrm{m}\, z < 0\}$, the space of holomorphic functions $\mathbf{H^2}(\mathbb{C}_-)$ is defined by:*

$$\mathbf{H^2}(\mathbb{C}_-) = \left\{ u; u \text{ holomorphic in } \mathbb{C}_-, \sup_{y<0} \int_{-\infty}^{+\infty} |u(x+iy)|^2 dx < +\infty \right\}.$$

In $\mathbb{R}_+^* \times \omega = \Omega \subset \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, the weighted L^2 space $L^{2,\gamma}(\Omega)$ is defined by $L^{2,\gamma}(\Omega) = L^2(\Omega, |x|^{-2\gamma}\, dx)$.

The connection between the two above spaces can be made via the Mellin transform with the help of Paley-Wiener theorem (see [23]).

Proposition 4.3. *1. In dimension 1, the Mellin transform*

$$M : L^2(]0,1], r^{-2\gamma}\frac{dr}{r}) \longrightarrow \mathbf{H^2}(\mathbb{C}_- + i\gamma)$$

is an isometry.

2. In the sectorial domain $\Omega = \mathbb{R}_+^* \times \omega$, the Mellin transform in the radial variable still denoted by M:

$$M : L^2\Big(\Omega \cap \{r \le 1\}, |x|^{-2\gamma} dx\Big) \longrightarrow \mathbf{H^2}\Big(\mathbb{C}_- + i(\gamma - \frac{n}{2}); L^2(\omega)\Big)$$

is an isometry.

4.2 The Sobolev spaces $H^{s,\gamma}$.

The general definition of the weighted Sobolev spaces $H^{s,\gamma}(\Omega)$, $\Omega = \mathbb{R}_+^* \times \omega$ is conveniently written in dimension $n > 1$ after introducing the Laplace-Beltrami operator Δ_S on S^{n-1}. The operator $\Lambda = (1 - \Delta_S)^{\frac{1}{2}}$, is a self-adjoint operator bounded from below by 1. Note that for $\lambda \geq 1$ the function $z \to (\lambda + i(z - i(\gamma - \frac{n}{2})))^s$ admits a holomorphic realization in $\{z, \mathcal{I}\mathrm{m}(z) < \gamma - \frac{n}{2}\}$ for any $s \in \mathbb{R}$.

Definition 4.4. *For $s \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$, a distribution $u \in \mathcal{D}'(\Omega)$ belongs to $H^{s,\gamma}(\Omega)$ if its radial Mellin transform Mu is the restriction to $[\mathbb{R} + i(\gamma - \frac{n}{2})] \times \omega$ of a function $v(z,\theta)$ on $\mathbb{R} \times S^{n-1}$ checking*

$$\left(\Lambda + i(z - i(\gamma - \frac{n}{2}))\right)^s v \in L^2(\mathbb{R} + i(\gamma - \frac{n}{2}); L^2(S^{n-1}))\,.$$

The support condition $\operatorname{supp} u \subset \{r \leq 1\}$ again has a simple translation due to Paley-Wiener theorem.

Proposition 4.5. *A distribution $u \in \mathcal{D}'(\Omega)$ satisfies $u \in H^{s,\gamma}(\mathbb{R}_+^* \times \omega)$ with $\operatorname{supp} u \subset \{r \leq 1\}$ if and only if its Mellin transform Mu is the restriction to $[\mathbb{R} + i(\gamma - \frac{n}{2})] \times \omega$ of a function $v(z,\theta)$ on $\mathbb{R} \times S^{n-1}$ checking*

$$\left(\Lambda + i(z - i(\gamma - \frac{n}{2}))\right)^s v \in \mathbf{H^2}(\mathbb{C}_- + i(\gamma - \frac{n}{2}); L^2(S^{n-1}))\,.$$

In the case when $s \in \mathbb{N}$, the definition of spaces $H^{s,\gamma}$ can be given in a simpler form:

$$\begin{aligned}
H^{s,\gamma} &= \{u \in L^2_{loc}(\Omega), (r\partial_r)^{\alpha_1}(\partial_\theta)^{\alpha_2} u \in L^2(\Omega, r^{-2\gamma}dx) = L^{2,\gamma}; \alpha_1 + |\alpha_2| \leq s\} \\
&= \{u \in L^2_{loc}(\Omega), \quad r^{|\alpha|} D^\alpha u \in L^{2,\gamma}\}, \quad D = \frac{1}{i}(\partial_r, \frac{1}{r}\partial_\theta), \\
\|u\|^2_{H^{s,\gamma}} &= \sum_{\alpha_1 + |\alpha_2| \leq s} \|(r\partial_r)^{\alpha_1} \partial_\theta^{\alpha_2} u\|^2_{L^{2,\gamma}}\,, \quad (s \in \mathbb{N})\,.
\end{aligned}$$

Remark 4.6. *Although the notations are slightly different, the spaces $\overset{\circ}{W}{}^m_\alpha$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, of Kondratiev in [13] and their generalization H^s_γ introduced by Dauge in [5] are among our spaces $H^{s,\gamma}$.*

1. *For $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we have $\overset{\circ}{W}{}^m_\alpha(\Omega) = H^m_{\alpha/2}(\Omega)$.*

2. *The relationship between our spaces $H^{s,\gamma}(\Omega)$ and the spaces $H^s_\gamma(\Omega)$ in [5] in the case $s \in \mathbb{N}$ can be obtained by induction on s owing to $(r\partial_r)(r^\gamma u) = \gamma r^\gamma u + r^\gamma (r\partial_r)u$. The general result is then obtained by complex interpolation and states $H^{s,\gamma}(\Omega) = H^s_{s-\gamma}(\Omega)$.*

3. *In particular, notice the relationships*

$$L^{2,\gamma} = H^{0,\gamma}, \quad L^{2,0} = L^2, \quad u \in L^{2,\gamma} \Leftrightarrow u \in H^0_{-\gamma}\,.$$

4.3 Link with the usual Sobolev spaces.

We work again in $\Omega = \mathbb{R}^*_+ \times \omega$ where ω is a regular domain in S^{n-1}. We are mainly interested in the behavior of functions in the neighborhood of the corner $r = 0$. Therefore, we focus on functions supported in $\{r \le 1\}$. With this support condition, the usual H^s, $s \ge 0$ regularity can be interpreted in terms of meromorphic Mellin transforms with the help of the weighted Sobolev spaces introduced before.

Proposition 4.7. *For every $s \in \mathbb{R}_+$ such that $s - \frac{n}{2} \notin \mathbb{N}$, we have the following properties. If u belongs to $H^s(\Omega)$ and has a support included in $\{r \le 1\}$ then for all $\alpha' \in \mathbb{N}^{n-1}$, $|\alpha'| \le [s]$, $M(\nabla_\theta^{\alpha'} u)$ admits a meromorphic extension in the complex half-plane $\left\{\mathcal{I}\mathrm{m}(z) < s - \frac{n}{2}\right\}$. Its poles are simple poles at $z_k = ik$, $|\alpha'| \le k \le [s - \frac{n}{2}]$ with the residues*

$$Res(M(\nabla_\theta^{\alpha'} u), ik) = -\frac{i}{(k - |\alpha'|)!}\partial_r^{k-|\alpha'|}(\frac{1}{r}\nabla_\theta)^{\alpha'} u(0)\,.$$

Moreover if $P_{u,[s-\frac{n}{2}]}$ denotes the Taylor expansion up to the order $[s - \frac{n}{2}]$ and if $\chi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ with $\operatorname{supp}\chi \subset \{r \le 1\}$ and $\chi \equiv 1$ in a neighborhood of $r = 0$, then the remainder $u - \chi(r)P_{u,[s-n/2]}$ satisfies

$$u - \chi(r)P_{u,[s-\frac{n}{2}]} \in H^{s,s}(\Omega)\,.$$

Remark 4.8.

1. *The Proposition 4.7 is given with the notation $H^{s,s}$, which correspond to H^s_0 in [5] (see Proposition AA.29 p 228).*

2. *In dimension n, the case $s - \frac{n}{2} \in \mathbb{N}$ is more delicate. We will always avoid these critical situations while working under the assumption of a little more or a little less regularity. As we will see it in Section 6, the idea is to work on the asymptotic types which generalize the Taylor expansion. We are mainly interested in the position of the poles which are here at $z = it_k$ with $t_k \left\{0, 1, \dots [s - \frac{n}{2}]\right\}$.*

5 A summary of Kondratiev theory and framework.

After reviewing some results of Kondratiev, the definition of the generalized asymptotic expansion is specified. This notion will be used in the presentation of our strategy in the next section.

5.1 Kondratiev results.

We adapt to our specific 2-dimensional framework with second order differential operators and with our notations, the general result of Kondratiev in [13]. We recall [2] that $\Omega = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ or $\Omega = \mathbb{R}^2$.

The second order differential operator reads

$$L(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$$

with coefficients $a_\alpha \in \mathcal{C}^\infty(\overline{\Omega})$. The operators involved in the boundary conditions are given by

$$B_\pm(x, \partial_x) = \sum_{|\beta| \leq m_\pm} \frac{b_\beta(x)}{|x|^{\mu_\beta}} \partial_x^\beta$$

with $\mu_\beta \leq m_\pm - |\beta|$ and we assume that everywhere the coefficients of the operators $B_\pm$ are infinitely differentiable.

We consider the problem

$$\left\{ \begin{array}{rl} L(x, \partial_x) u & = f \\ B_-(x, \partial_x) u \Big|_{\theta = \theta_-} & = g_- \\ B_+(x, \partial_x) u \Big|_{\theta = \theta_+} & = g_+ \end{array} \right. \tag{16}$$

It is assumed that this boundary value problem satisfies the Lopatinskii conditions along $\partial\Omega \setminus O$. This is true for example with the boundary conditions (6)(7).

We denote by L_0, $B_{0,\pm}$ the principal parts of the operators L, $B_\pm$ with coefficients fixed at the origin. We write then the operators L_0, $B_{0,\pm}$ in polar coordinate

$$L_0(x, \partial_x) = r^{-2} \tilde{L}_0(r\partial_r, \partial_\theta), \quad B_{0,\pm}(x, \partial_x) = r^{-m_\pm} \tilde{B}_{0,\pm}(r\partial_r, \partial_\theta).$$

[2] Actually these general results will be applied in the full domain Ω and in the subdomains Ω_1 and Ω_2. The notation $\theta_\pm$ for the possible limiting angles actually refers to Ω_1 in our presentation of the domain decomposition algorithm.

According to the analytic Fredholm theory (see for example [23]), the system

$$\begin{cases} \tilde{L}_0(iz,\partial_\theta)v &= \varphi \\ \tilde{B}_{0,-}(iz,\partial_\theta)v(\theta_-) &= \gamma_- \\ \tilde{B}_{0,+}(iz,\partial_\theta)v(\theta_+) &= \gamma_+ \end{cases} \tag{17}$$

admits a meromorphic resolvent denoted by $\mathcal{R}(z)$ such that for given $(\varphi,\gamma_-,\gamma_+)$ the solution $v = v(z)$ equals

$$v(z) = \mathcal{R}(z)(\varphi,\gamma_-,\gamma_+)$$

as an $L^2((\theta_-,\theta_+))$-valued meromorphic function of $z \in \mathbb{C}$.

The poles of the resolvent $\mathcal{R}(z)$ for the principal part and its residues play an important role in Kondratiev theory.

The first result is concerned with the case when the operators $L(x,\partial_x)$ and $B_\pm(x,\partial_x)$ are homogeneous operators.

Proposition 5.1. *Assume that $u \in H^{k+2,k+2-\frac{\alpha}{2}}(\Omega)$ solves (16) with L replaced by L_0, $B_\pm$ replaced by $B_{0,\pm}$, with $f \in H^{k_1,k_1-\frac{\alpha_1}{2}}(\Omega)$ and $g_\pm \in H^{k_1+2-m_\pm-\frac{1}{2},k_1+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}(\mathbb{R}_+^*)$. Assume moreover the condition*

$$h_1 = \frac{2k_1+4-2-\alpha_1}{2} > \frac{2k+4-2-\alpha}{2} = h, \quad k_1 \geq k$$

and that the resolvent $\mathcal{R}(z)$ of the problem (17) has no poles on the straight line $\mathcal{I}\mathrm{m}(z) = h_1$.

Then u satisfies

$$u = \sum_j \sum_{\nu=0}^{\mu_j-1} a_{j,\nu} r^{-i\lambda_j} \ln^\nu(r) \psi_{\nu,j}(\theta) + w,$$

where

- *$\psi_{\nu,j}$ are infinitely differentiable functions independent of $u(x)$, and λ_j are poles of multiplicity μ_j of the function $\mathcal{R}(z)$ such that $h < \mathcal{I}\mathrm{m}(\lambda_j) < h_1$;*
- *the remainder w belongs to $H^{k_1+2,k_1+2-\frac{\alpha_1}{2}}$ with*

$$\|w\|_{H^{k_1+2,k_1+2-\frac{\alpha_1}{2}}} \leq C\Big[\|u\|_{H^{k+2,k+2-\frac{\alpha}{2}}} + \|f\|_{H^{k_1,k_1-\frac{\alpha_1}{2}}} + \|g_\pm\|_{H^{k_1+2-m_\pm-\frac{1}{2},k_1+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}}\Big].$$

In the second statement which is for general L and $B_\pm$ in (16), there are additional constraint on the pairs (α,k) and (α_1,k_1). Contrary to the homogeneous case, it is not possible to get an asymptotic expansion at any order by simply looking at the principal part of the operators. Nevertheless the study of the principal part provides the first terms of the asymptotic expansion, which the meaning on the limitation on (α, k). For the sake of simplicity the result is stated with $k_1 = k$.

Proposition 5.2. *Assume that $u \in H^{k+2,k+2-\frac{\alpha}{2}}(\Omega)$, solves (16) with $f \in H^{k,k-\frac{\alpha_1}{2}}(\Omega)$ and $g_\pm \in H^{k+2-m_\pm-\frac{1}{2},k+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}(\mathbb{R}_+^*)$, $\alpha-2 \le \alpha_1 < \alpha$, and with $\operatorname{supp} f$ and $\operatorname{supp} g_\pm$ contained in $\{r \le \rho_0\}$, $\varrho_0 > 0$. Assume moreover that the resolvent $\mathcal{R}(z)$ of the principal part (17) has no pole on the straight line $h_1 = \frac{-2-\alpha_1+2k+4}{2}$. Then the solution $u(x)$ has the form*

$$u = \sum_j \sum_{\nu=0}^{\mu_j-1} a_{j,\nu} r^{-i\lambda_j} \ln^\nu(r) \psi_{\nu,j}(\theta) + w \tag{18}$$

where λ_j are the poles of $\mathcal{R}(z)$ contained in the strip $\{h < \mathcal{I}\mathrm{m}\, z < h_1\}$, $h = \frac{-2-\alpha+2k+4}{2}$, μ_j is the multiplicity of λ_j and w belongs to $H^{k+2,k+2-\frac{\alpha_1}{2}}$ with the inequality

$$\|w\|_{H^{k+2,k+2-\frac{\alpha_1}{2}}} \le C\Big[\|u\|_{H^{k+2,k+2-\frac{\alpha}{2}}} + \|f\|_{H^{k,k-\frac{\alpha_1}{2}}} + \|g_\pm\|_{H^{k+2-m_\pm-\frac{1}{2},k+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}}\Big].$$

Remark 5.3. *In [13] these results are stated in weighted spaces denoted by $\overset{\circ}{W}{}^m_\alpha$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. With our notations, they coincide with $H^{m,m-\frac{\alpha}{2}}$.*

5.2 An example.

Here is a classical example which will motivate our general definition.

Consider first in $\Omega = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ [3] a solution $u \in H^1(\Omega)$ such that $\operatorname{supp} u \subset \{r \le 1\}$, to the homogeneous boundary value problem:

$$\begin{cases} -\Delta u = f & \in L^{2,\gamma}(\Omega) \\ u(r,\theta_\pm) = g_\pm(r) & \in L^{2,\gamma+2-\frac{1}{2}}(\mathbb{R}_+^*)\,. \end{cases} \tag{19}$$

The conditions on u implies that its Mellin transform is holomorphic in $\{\mathcal{I}\mathrm{m}\, z < 0\}$. Since the support conditions are transmitted to $f, g_\pm$, the Mellin transforms of f and $g_\pm$ belong respectively to $\mathbf{H}^2(\mathbb{C}_- + i(\gamma-1); L^2(\theta_-,\theta_+))$ and $\mathbf{H}^2(\mathbb{C}_- + i(\gamma+1))$ according to Proposition

[3] In the domain decomposition, the notations (θ_-, θ_+) are used for the subdomain Ω_1. Here they are convenient for a general sector Ω.

4.3. We focus here on the case $\gamma > -1$. After taking the radial Mellin transform, the equation becomes

$$\begin{cases} \left[\partial_\theta^2 - z^2\right](Mu)(z,\theta) = M[-r^2 f](z,\theta) = -(Mf)(z-2i,\theta) \\ Mu(z,\theta_\pm) = Mg_\pm(z,\theta_\pm)\,. \end{cases} \tag{20}$$

By introducing the resolvent $\mathcal{R}(z)$, the solution writes:

$$Mu(z) = \mathcal{R}(z)[-Mf(z-2i), Mg_-(z), Mg_+(z)]\,. \tag{21}$$

The resolvent $\mathcal{R}(z)$ is an operator valued meromorphic function which can be computed explicitly here. One can determine $\mathcal{R}(z)$ by using the classical theory of ordinary differential equations:

$$\begin{aligned} \mathcal{R}(z)[\varphi,\gamma_-,\gamma_+] \quad &= \frac{\sinh(z(\theta-\theta_-))}{\sinh(z(\theta_+-\theta_-))}\gamma_+ - \frac{\sinh(z(\theta-\theta_+))}{\sinh(z(\theta_+-\theta_-))}\gamma_- \\ &+ \frac{\sinh(z(\theta_- -\theta))}{z\sinh(z(\theta_+-\theta_-))}\int_{\theta_-}^{\theta_+}\sinh(z(\theta_+-s))\varphi(s)ds \\ &+ \frac{1}{z}\int_{\theta_-}^{\theta}\sinh(z(\theta-s))\varphi(s)ds\,. \end{aligned}$$

Here $\varphi(z,\theta) = -Mf(z-2i,\theta), \gamma_-(z) = Mg_-(z,\theta_-)$ and $\gamma_+(z) = Mg_+(z,\theta_+)\,.$

It has simple poles at $z = in\frac{\pi}{\theta_+-\theta_-}$, for $n \in \mathbb{Z}$.

Taking into account the a priori information on u, f and $g_\pm$, we conclude that the Mellin transform Mu is an $L^2(\theta_-,\theta_+)$-valued meromorphic function in $\{\mathcal{I}\mathrm{m}\, z < \gamma+1\}$. By assuming $\gamma+1>0$ and $\gamma+1 \notin \left\{\frac{n\pi}{\theta_+-\theta_-},\ n\in\mathbb{N}\right\}$ it may have simple poles at the points

$in\frac{\pi}{\theta_+-\theta_-}$, $\frac{n\pi}{\theta_+-\theta_-} < (\gamma+1)$, $n\in\mathbb{N}$, with:

$$Res(Mu, in\frac{\pi}{\theta_+-\theta_-}) = c_{n,u}\sin(n\frac{\pi}{\theta_+-\theta_-}(\theta-\theta_-)), \quad \frac{n\pi}{\theta_+-\theta_-} < \gamma+1\,.$$

Actually the poles $i\frac{n\pi}{\theta_+-\theta_-}$ with negative $n<0$ are eliminated by the condition $u\in H^1(\Omega)$, $\operatorname{supp} u \subset \{r\le 1\}$. One recovers the result of Proposition 5.1:

$$u = \chi(r) \sum_{n\in\mathbb{N},\ n<\frac{(\theta_+-\theta_-)(\gamma+1)}{\pi}} ic_{n,u} r^{\frac{n\pi}{\theta_+-\theta_-}} \sin\left(\frac{n\pi}{\theta_+-\theta_-}(\theta-\theta_-)\right) + R$$

with $\chi\in\mathcal{C}_0^\infty(\mathbb{R}_+)$, $\operatorname{supp}\chi\subset\{r\le 1\}$, $\chi\equiv 1$ in the neighborhood of 0, and $R\in H^{2,\gamma+2}(\Omega)\subset L^{2,\gamma+2}(\Omega)$.

We started with data $f \in L^{2,\gamma}(\Omega)$ and $g_\pm \in L^{2,\gamma+3/2}(\mathbb{R}_+^*)$ so that their Mellin transforms is holomorphic in $\{\mathcal{I}\mathrm{m}\, z < \gamma - 1\}$ and $\{\mathcal{I}\mathrm{m}\, z < \gamma + 1\}$. For $\gamma > -1$ another interesting case permitted by the assumption $u \in H^1(\Omega)$, occurs when the Mellin transforms $Mf(z - 2i)$ and $Mg_\pm$ have poles in $\{0 \leq \mathcal{I}\mathrm{m}\, z < \gamma + 1\}$. Then equation (21) implies

$$\{\text{poles } (u)\} \subset \{ \text{ poles } (\mathcal{R}(z))\} \cup \{ \text{ poles } (-Mf(z-2i), Mg_\pm)\}\,. \tag{22}$$

We will discuss later general consequences in domain decomposition methods (see Section 6). A specific case enlightens the limitation on the asymptotic expansion in non homogeneous boundary value problems. Consider now the boundary value problem

$$\begin{cases} \eta u - \Delta u = f & \in L^{2,\gamma}(\Omega) \\ u(r,\theta_\pm) = g_\pm(r) & \in L^{2,\gamma+2-\frac{1}{2}}(\mathbb{R}_+^*)\,, \end{cases}$$

with $\eta > 0$. We assume again that the (unique) solution $u \in H^1(\Omega)$ is supported in $\{r \leq 1\}$ and the number $\gamma > -1$ can be assumed to be a large positive number. After putting $-\eta u$ with f in the right-hand side, equation (21) now implies

$$Mu(z) = \mathcal{R}(z)\,[M[-f + \eta u](z - 2i), Mg_-, Mg_+]\;.$$

Hence the poles of Mu in $\{0 \leq \mathcal{I}\mathrm{m}\, z < 2\}$ come only from the poles of the resolvent $\mathcal{R}(z)$ while for $\mathcal{I}\mathrm{m}\, z \geq 2$ the poles of $Mu(z - 2i)$ have to be added to the one of $\mathcal{R}(z)$. Only the first terms of the asymptotic expansion can be deduced from the poles of $\mathcal{R}(z)$. This is a reason to the limitation $\alpha - 2 \leq \alpha_1 < \alpha$ in Proposition 5.2.

5.3 Generalization of the Taylor expansion.

The final aim of our work is to improve the convergence of domain decomposition methods in the neighborhood of corners. The relevant analysis has to be done in a neighborhood of $r = 0$, with several consequences:

1. First, it is sufficient to consider truncated solutions $\chi(r)u$, with $\chi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$, $\mathrm{supp}\,\chi \subset \{r \leq 1\}$ and $\chi \equiv 1$ in a neighborhood of $\{r = 0\}$. One can even consider directly the the situation $u = \chi(r)u$. The variational formulation of Section 3 ensures that the solution to a subdomain problem also satisfies $u \in H^1(\Omega)$. Hence, the radial Mellin transform $M[u] = M[\chi u]$ is holomorphic in the half-plane $\{\mathcal{I}\mathrm{m}\, z < 0\}$. Since $L(x, \partial_x)$

and $B_\pm(x,\partial_x)$ are differential operators, the initial data $f \in L^2(\Omega)$ (resp. $g_\pm$) have a holomorphic Mellin transform in $\{\mathcal{I}\mathrm{m}\, z < -1\}$ (resp. $\{\mathcal{I}\mathrm{m}\, z < 0\}$).

2. We are interested in the asymptotic expansion as $r \to 0$ and all the interesting information is contained in the exponent γ while the exponent s concerns an additional radial regularity (with respect to the characteristic vector field $r\partial_r$). The next notion will be reduced to the case $s = 0$.

3. According to (22), the poles of the Mellin transform of the solution can also be produced by the ones of the data f, $g_\pm$. This occurs even with regular data since these poles encode the Taylor expansion according to Proposition 4.7.

The next definition gathers all these remarks by considering only Mellin transforms which are meromorphic in $\{\mathcal{I}\mathrm{m}\, z < \gamma\}$. It permits to treat with the same language the Taylor expansion of regular functions and the general asymptotic expansion given by Kondratiev theory.

Definition 5.4. *An asymptotic type in* $\Omega = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ [4] *(resp. on* $\mathbb{R}_+^*$*) is a finite collection* $\mathcal{T} = \left((\lambda_j, \mu_j, \varphi_{j,\nu}(\theta),\ 0 \le \nu \le \mu_j - 1)\right)_{j\in\{1,\ldots,N\}}$ *(resp.* $\mathcal{T} = \left((\lambda_j, \mu_j)\right)_{j\in\{1,\ldots,N\}}$*) where the* λ_j *are points of* $\{\mathcal{I}\mathrm{m}\, z \ge 0\}$*,* μ_j *is the multiplicity of the pole* λ_j *and* $\varphi_{j,\nu} \in L^2((\theta_-, \theta_+))$*. For* $\gamma \ge 0$*,* $L^{2,\gamma}_{r\le 1,\mathcal{T}}(\Omega)$ *(resp.* $L^{2,\gamma}_{r\le 1,\mathcal{T}}(\mathbb{R}_+^*)$*) will denote the set of the functions* $u \in L^2(\Omega)$ *(resp.* $u \in L^2(\mathbb{R}_+^*)$*) with support in* $\{r \le 1\}$ *whose Mellin transform admits a meromorphic extension in* $\{\mathcal{I}\mathrm{m}\, z < \gamma - 1\}$ *(resp* $\left\{\mathcal{I}\mathrm{m}\, z < \gamma - \frac{1}{2}\right\}$*) with a trace in* $L^2(\mathbb{R} + i(\gamma - 1); L^2(\theta_-, \theta_+))$ *(resp.* $L^2(\mathbb{R} + i(\gamma - \frac{1}{2}))$*), with poles and residues given by the asymptotic type* $\mathcal{T}$*.*

A simple elimination of the poles provides the writing

$$u(x) = 1_{[0,1]}(r) \sum_{0 \le \mathcal{I}\mathrm{m}\,\lambda_j < \gamma - 1} \sum_{\nu=0}^{\mu_j - 1} c_{j,\nu} r^{-i\lambda_j} \ln^\nu(r) \varphi_{j,\nu}(\theta) + w(x)\,, \tag{23}$$

with $w \in L^{2,\gamma}(\Omega)$, for any $u \in L^{2,\gamma}_{r\ge 1,\mathcal{T}}(\Omega)$.

6 Strategy.

The previous Section can be summarized by stating that the asymptotic type, introduced in Definition 5.4, of the solution to a second order elliptic problem in a conical domain, depends

[4] possibly $\Omega = \mathbb{R}_+^* \times S^1$

on the interior operator, the geometry (angle) of the domain, the boundary operator and the right-hand side.

The first observation concerned with the domain decomposition is that the asymptotic types associated with the complete domain Ω boundary value problem has nothing to do with the ones of the subdomain problems. After some numerical observations, it is considered here and in [21] as the main (local) obstruction to a rapid convergence.

The second observation is that like in the case of smooth interfaces our class of interface boundary conditions (7) permits some flexibility in the choice of the coefficients $\alpha_\pm, \beta_\pm$. Hence the question is whether in a subdomain Ω_1 there is a choice of pairs $(\alpha_\pm, \beta_\pm)$ which provides the best possible matching with the asymptotic type of the solution to the complete domain problem.

As a third observation, notice that at each step of the domain decomposition algorithm (9) (10), every subdomain problem as a nonzero right-hand side concentrated in the boundary conditions. According to the general principle which says that the poles of the Mellin transform Mu of the solution are produced by the poles of the resolvent and the poles of the right-hand side, any new iteration n in the domain decomposition may produce new poles or increase their multiplicity. Following completely the propagation of those poles through the algorithm seems a complex task especially for more than 2 subdomains.

Starting from those three observations, our approach consists in determining the "best pairs" $(\alpha_\pm, \beta_\pm)$ from necessary conditions and then hope that in practice this choice which cannot be worse than the arbitrary one improves significantly the speed of convergence. This last point will be shown in the subsequent numerical article. The general principle which leads the next calculations is the following : *Assume that at step n the error functions $e_j^n = u_j^n - u$, $j = 1, \ldots, N$, have the asymptotic type of the full domain Ω problem, then the error functions e_j^{n+1} should keep this asymptotic type up to some large enough order as* $r \to 0$. The better the matching is, the faster the convergence is expected.

We will work in spaces $L_{\mathcal{T}}^{2,\gamma}$ with $0 < \gamma < 2$ by taking γ as large as possible so that the asymptotic types $\mathcal{T}$ coincide in the complete domain Ω and in the subdomains Ω_j, $j = 1, \ldots, N$. When the subdomain boundary problem (9), or its principal part (10), is solved artificial poles occur in the Mellin transform of e_i^{n+1}. The analysis focuses on the first artificial poles, which is the one with the smallest imaginary part and corresponds to the biggest factor $r^{-i\lambda_j}$ in (23). The corresponding residue depends on the right-hand side in (9) or (10) which at first order is given by the asymptotic type of the full domain problem. The question now becomes: Is it possible to choose the pairs $(\alpha_\pm, \beta_\pm)$ in Ω_1 so that if the data involved in the right-hand side of (9)(10) have the right asymptotic type, then the first artificial poles is canceled by a vanishing residue. Actually the computation will show that such a choice of the pairs is not always possible and we finally have to choose between two ways for every subdomain Ω_j, $j = 1, \ldots, N$:

1. First check if it is possible to cancel the first artificial pole according to the previous process.

2. If the first approach has no solution, choose the pairs $(\alpha_\pm, \beta_\pm)$ so that the first artificial pole has the largest possible imaginary part.

In the first case, the best exponent γ will be given by the imaginary part of the second artificial pole. In the second case, it will be given by the imaginary part of the first artificial pole pushed as far as possible from 0. After taking the minimal γ over all the subdomains Ω_j, $j = 1, \ldots, N$, this provides an idea of the accuracy of the approximation by the domain decomposition method around the corner.

We end this general presentation by two remarks.

1. We consider here differential operators with real coefficients. Hence the principal part of the asymptotic expansion (18) must be real for real data. This implies that the position of the poles of the resolvent $\mathcal{R}(z)$ must be symmetric with respect to the imaginary axis $i\mathbb{R}$. Actually, we check here in every cases that the poles of $\mathcal{R}(z)$ lie exactly on the imaginary axis.

2. The variational formulation ensures that the solution belongs to H^1 for the complete problem or for the subdomain problems. Hence a multiple pole at $z = 0$ is not possible because $r \ln^k(r) \notin H^1$ in dimension 2 for $k > 0$. The simple pole at $z = 0$ corresponds to the possibly non zero value of the solution $u(O)$ when $u \in H^{1+\delta}$ with $\delta > 0$.

Those two remarks suggest that we have to focus on the analysis of the poles $z = it$ with $t > 0$. This will be checked with details in the next Section.

7 Theoretical determination of good pairs $(\alpha_\pm, \beta_\pm)$ in a subdomain for a model problem.

The strategy presented in Section 6 is now implemented in some specific cases.
The domain decomposition methods generates two types of corners: There are corners which are in the interior of the global domain and other corners which are on the edge of the global domain. The treatment of these two categories of corners is slightly different. The choice of interface conditions takes into account the nature of the corners, by imposing $\beta_\pm = 0$ or by permitting any $\beta_\pm > 0$. For the corners in the interior of the global domain, or on the edge of the global domain with Neumann boundary condition, the coefficient $\beta_\pm$ has to be 0 in order to make possible a non vanishing value $u(O) \neq 0$ (see Section 3). Meanwhile $\beta_\pm \neq 0$ is possible when the corner is on the edge of the global domain with Dirichlet condition (see Section 3). For this reason, the two types of corner will be treated separately in the optimizing process of the coefficients $(\alpha_\pm, \beta_\pm)$.

7.1 Improved interface conditions for domain decomposition of non-convex polygonal domain.

The sector $\Omega = \{(r\cos\theta, r\sin\theta), r > 0, \theta_0 < \theta < \theta_+\}$, with $\theta_+ - \theta_0 \in (0, 2\pi)$, in polar coordinates is decomposed into $\Omega_1 = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ and $\Omega_2 = \mathbb{R}_+^* \times (\theta_0, \theta_-)$, with $\theta_0 < \theta_- < \theta_+$(see Figure 1).

By considering Dirichlet problem in Ω, we carry out the general strategy (Section 6) with details. We find an explicit linear relation which has to be satisfied by the pair (α_-, β_-) in order to permit the matching of the asymptotic types.

7.1.1 The Dirichlet problem.

a) Asymptotic type in Ω.

Consider the homogeneous Dirichlet problem

$$(\eta - \Delta)u = f \quad u\big|_{\theta=\theta_0} \equiv 0 \quad u\big|_{\theta=\theta_+} \equiv 0\,, \tag{24}$$

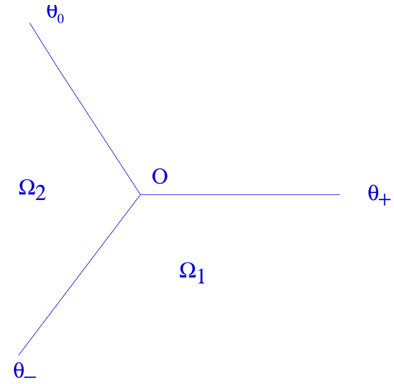

Figure 1: Two domain decomposition.

which is well posed in $H^1(\Omega)$ for $f \in L^2(\Omega)$. Assume that for an approximate solution v, the consistency is well satisfied around $r = 0$. Namely, by setting $e = v - u$,

$$(\eta - \Delta)e = \rho \quad e\big|_{\theta=\theta_0} \equiv 0 \quad e\big|_{\theta=\theta_+} \equiv 0,$$

with $\rho(r,\theta) = O(r^\infty)$. Then Proposition 5.2 says that even in this case the conclusion is

$$e(r,\theta) = a_1 r^{\frac{1}{x_0}} \sin\left(\frac{\theta-\theta_0}{x_0}\right) + o(r^{\frac{1}{x_0}}), \quad a_1 \in \mathbb{R}, \tag{25}$$

with $x_0 = \frac{\theta_+ - \theta_0}{\pi}$. According to Definition 5.4, the asymptotic type in Ω for $\gamma < \frac{2}{x_0}$ is $\mathcal{T} = (\frac{i}{x_0}, 1, \sin(\frac{\theta-\theta_0}{x_0}))$. It describes the first term in the asymptotic expansion u to (24) with a right-hand side f rapidly vanishing as $r \to 0$.

b) Subproblem in Ω_1.

We focus on the subdomain Ω_1. The treatment of Ω_2 is similar. The boundary problem (1) in Ω_1 with the interface conditions (6) solved by the error $e_1^{n+1} = u_1^{n+1} - u$ reads

$$\left\{\begin{array}{rl} \Big(\eta - \frac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2)\Big)e_1^{n+1}(r,\theta) & = 0 \\ e_1^{n+1}(r,\theta_+) & = 0 \\ \Big(-\frac{1}{r}\partial_\theta + \tilde\beta_-(r) - \frac{1}{2}\partial_r(\tilde\alpha_-(r)\partial_r)\Big)e_1^{n+1}(r,\theta_-) & = g_-(r) \end{array}\right. \tag{26}$$

where $g_-(r) = \Big(-\frac{1}{r}\partial_\theta + \tilde\beta_-(r) - \frac{1}{2}\partial_r(\tilde\alpha_-(r)\partial_r)\Big)e_2^n(r,\theta_-)$. This problem admits a well posed variational formulation in a subspace of $H^1(\Omega_1)$ with the sign conditions $\alpha_-, \beta_- > 0$. Remind that $\tilde\alpha_-$ and $\tilde\beta_-$ are chosen so that variational problem is well posed according to (7) and Section 3. Following Kondratiev (see Section 5), we are led to consider the z-dependent problem derived from the principal part

$$(\partial_\theta^2 - z^2)\widehat{e}_1(z,\theta) = 0 \qquad \widehat{e}_1(z,\theta_+) = 0 \qquad (\partial_\theta - \beta_- - \frac{\alpha_-}{2}z^2)\widehat{e}_1(z,\theta_-) = \widehat{g}_-(z)\,, \tag{27}$$

whose solution is

$$
\begin{aligned}
& a(z)e^{z(\theta-\theta_-)} + b(z)e^{-z(\theta-\theta_+)} \\
& a(z) = \mathcal{R}(z)\widehat{g}_-(z),\ b(z) = -a(z)e^{z(\theta_+-\theta_-)}, \\
\text{with} \quad & \mathcal{R}(z) = \Big[(z-\beta_- - \frac{\alpha_-}{2}z^2) + (z+\beta_- + \frac{\alpha_-}{2}z^2)e^{2z(\theta_+-\theta_-)}\Big]^{-1}.
\end{aligned}
$$

We recall that $u \in H^1$ implies that the multiplicity of the pole of Mu at $z = 0$ is 1 and permits to focus on the poles with positive imaginary part.

Proposition 7.1. *The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are the purely imaginary complex numbers $z = it$, with $t > 0$ and*

$$
\tan(\pi x t) = \frac{2t}{\alpha_- t^2 - 2\beta_-}, \qquad x = \frac{\theta_+ - \theta_-}{\pi}, \tag{28}
$$

whose positive solutions are denoted by $t_k, k \in \mathbb{N}^*$ in the increasing order.

Proof. In order to determine the poles of $\mathcal{R}(z)$, it is enough to solve the equation

$$
(z-\beta_- - \frac{\alpha_-}{2}z^2) + (z+\beta_- + \frac{\alpha_-}{2}z^2)e^{2z(\theta_+-\theta_-)} = 0\,.
$$

Therefore the poles are the solutions to

$$
e^{2z(\theta_+-\theta_-)} = \frac{-z+\beta_- + \frac{\alpha_-}{2}z^2}{z+\beta_- + \frac{\alpha_-}{2}z^2}\,. \tag{29}
$$

As $z \in \mathbb{C}$, we note $z = r + it$ with $r, t \in \mathbb{R}$. Taking the module of (29) we obtain

$$
e^{4r(\theta_+-\theta_-)} = \psi(r) \tag{30}
$$

where

$$
\psi(r) = \frac{(-r+\beta_- + \alpha_-\frac{r^2-t^2}{2})^2 + t^2(-1+\alpha_- r)^2}{(r+\beta_- + \alpha_-\frac{r^2-t^2}{2})^2 + t^2(1+\alpha_- r)^2}\,.
$$

The proof that $r = 0$ relies on the equality

$$
\psi(r) - 1 = \frac{-2r(2\beta_- + \alpha_-|z|^2)}{(r+\beta_- + \alpha_-\frac{r^2-t^2}{2})^2 + t^2(1+\alpha_- r)^2}\,.
$$

which implies $r(\psi(r)-1) < 0$.

Since $\theta_+ > \theta_-$ then we can deduce:

- If $r > 0$ we get

$$e^{4r(\theta_+ - \theta_-)} > 1 \text{ and } \psi(r) < 1$$

therefore one cannot have $r > 0$.

- If $r < 0$ we get

$$e^{4r(\theta_+ - \theta_-)} < 1 \text{ and } \psi(r) > 1$$

therefore we cannot have $r < 0$.

Therefore the equality (30) implies $r = 0$.

The poles with a positive imaginary part have the form $z = it$, with $t > 0$ and

$$e^{2it(\theta_+ - \theta_-)} = \frac{-it + \beta_- + \frac{\alpha_-}{2}(it)^2}{it + \beta_- + \frac{\alpha_-}{2}(it)^2}. \tag{31}$$

Writing the second member of (31) in the exponential form

$$e^{2it(\theta_+ - \theta_-)} = e^{i\pi} e^{-2i \arctan(\frac{\alpha_-}{2}t - \frac{\beta_-}{t})}, \tag{32}$$

we deduce

$$2t(\theta_+ - \theta_-) = \pi - 2\arctan(\frac{\alpha_- t^2 - 2\beta_-}{2t}) mod(2\pi).$$

Hence t satisfies (28). $\square$

The best way to study the 2 first positive solutions of (28) is the graphical representation of $t \longrightarrow \tan(\pi x t)$ and $t \longrightarrow \frac{2t}{\alpha_- t^2 - 2\beta_-}$. There are 2 different cases according to $\sqrt{\frac{2\beta_-}{\alpha_-}} < \frac{1}{2x}$ or $\sqrt{\frac{2\beta_-}{\alpha_-}} > \frac{1}{2x}$ which compares the positions of the first vertical asymptotes.

Below is the discussion between the two possible treatments of the first pole it_1 according to the general strategy presented in Section 6.

1. When the first pole (it_1) can be canceled for well prepared data (first approach), the first active artificial pole becomes it_2 with

$$\begin{aligned} t_2 &> \tfrac{1}{x} \quad \text{if } \sqrt{\tfrac{2\beta_-}{\alpha_-}} < \tfrac{1}{2x} \\ t_2 &> \tfrac{3}{2x} \quad \text{if } \sqrt{\tfrac{2\beta_-}{\alpha_-}} > \tfrac{3}{2x} \\ t_2 &> \tfrac{1}{x} \quad \text{if } \sqrt{\tfrac{2\beta_-}{\alpha_-}} > \tfrac{1}{2x}. \end{aligned}$$

2. The second approach consists in pushing as far as possible the first pole it_1 from $z = 0$. A good choice is $\sqrt{\frac{2\beta_-}{\alpha_-}} >> \frac{1}{2x}$ and $t_1 \simeq \frac{1}{x}$.

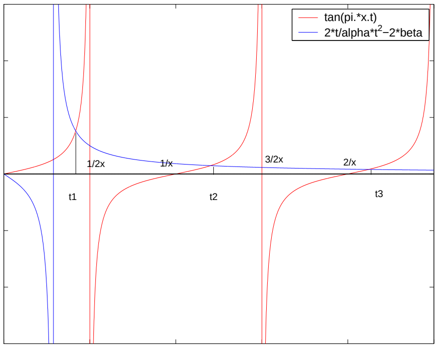

Figure 2: Case of $\sqrt{\frac{2\beta_-}{\alpha_-}} < \frac{1}{2x}$.

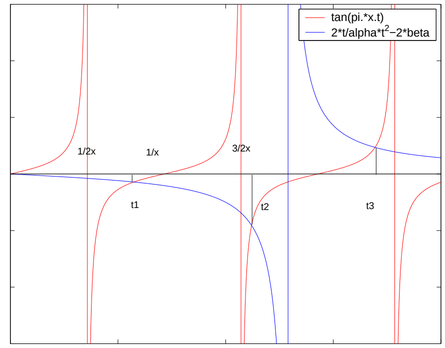

Figure 3: Case of $\sqrt{\frac{2\beta_-}{\alpha_-}} > \frac{1}{2x}$.

When it is possible the first approach with $t_2 > \frac{1}{x}$ is always better than the second one with $t_1 < \frac{1}{x}$.

c) Cancellation of the first artificial pole for well prepared data.

In the subdomain Ω_1 and for a general right-hand side in (26), the first artificial term in the asymptotic expansion of e_1^{n+1} appears with the factor r^{t_1}, with t_1 defined in Proposition 7.1. The first (and most efficient) approach assumes that at step n the error as the natural asymptotic type associated with the global problem:

$$e_2^n(r,\theta) = a_1 r^{\frac{1}{x_0}} \sin(\frac{\theta-\theta_0}{x_0}) + o(r^{\frac{1}{x_0}}). \tag{33}$$

With an additional truncation in $\{r \leq R\}$, we set like in the domain decomposition algorithm

$$g_-(r) = 1_{\{r\leq R\}}\Big(\partial_\theta - \beta_- + \frac{\alpha_-}{2}(r\partial_r)^2\Big) e_2^n(r,\theta_-)\,.$$

After forgetting the $o(r^{\frac{1}{x_0}})$ remainder and taking the Mellin transform, this provides

$$\widehat{g}_-(z) = \frac{R^{\frac{1}{x_0}+iz}}{\frac{1}{x_0}+iz} a_1 \left[\frac{1}{x_0}\cos(\frac{\theta_- - \theta_0}{x_0}) + (-\beta_- + \frac{\alpha_-}{2x_0^2})\sin(\frac{\theta_- - \theta_0}{x_0})\right]. \tag{34}$$

Here the cancellation of the first artificial pole it_1 is reduced to the simple condition

$$\widehat{g}_-(it_1) = 0 \tag{35}$$

Proposition 7.2. *The equation (35) is satisfied if and only if the pair* $(\alpha_-,\beta_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *satisfies*

$$-\beta_- + \frac{\alpha_-}{2x_0^2} = \frac{1}{x_0 \tan(\frac{\pi x}{x_0})}, \quad x = \frac{\theta_+ - \theta_-}{\pi}\,. \tag{36}$$

Proof. The equation (35) does not depend on the truncation parameter R and reads simply

$$\frac{1}{x_0}\cos(\frac{\theta_- - \theta_0}{x_0}) + (-\beta_- + \frac{\alpha_-}{2x_0^2})\sin(\frac{\theta_- - \theta_0}{x_0}) = 0\,.$$

This yields the result. □

It was checked in many cases that the system (8) (36) admits a solution $(\alpha_-, \beta_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ for many angles $0 < (\theta_+ - \theta_-) < (\theta_+ - \theta_0) < 2\pi$ and for various values of h. A general argument can be given by looking in the (α_-, β_-) plane at the polygonal line

$$\mathrm{PL}_h := \{(\alpha_-, \beta_-); \alpha_- = \frac{c_1 h^{3/4}}{\Phi(h)}, \beta_- \geq \frac{c_2\Phi(h)}{h^{1/4}}\} \cup \{(\alpha_-, \beta_-); \alpha_- \leq \frac{c_1 h^{3/4}}{\Phi(h)}, \beta_- = \frac{c_2\Phi(h)}{h^{1/4}}\}$$

for $c_1, c_2 \in \mathbb{R}_+^*$ fixed[5]. Our framework where $\Phi(h) = 3h$ or $\Phi(h) = o(h)$ as $h \to 0$ ensures that PL_h has a non empty intersection with the straight line (36) whose slope is $\frac{1}{2x_0^2} > 0$, for $h > 0$ small enough. Owing to $\frac{1}{\tan(\frac{\pi x}{x_0})} = 0$ for $x = \frac{x_0}{2}$, this discussion is even more convincing when x lies in a neighborhood of $\frac{x_0}{2}$. Theoretically, a solution to the system (8) (36) can be found as soon as the mesh size is small enough $h < h_\delta$ when $x \in [\delta x_0, (1-\delta)x_0]$ with $\delta > 0$.

Hence , the first approach can be implemented. It permits to expect after enough iterations a matching of the domain decomposition approximation and the full solution up to $O(r^{\min\{t_2(\Omega_2), t_2(\Omega_1)\}})$. The notation $t_2(\Omega_j)$ refers to the second positive solution to (28) adapted to the domain Ω_j, $j = 1, 2$.

7.1.2 The Neumann problem.

Consider now the Neumann problem with $\partial_n u = 0$ on $\partial\Omega$. In order to have a well posed problem in Ω_1 and Ω_2 which permits non null values at the corner, we take $\beta_- = 0$.

a) Asymptotic type in Ω.

Consider the Neumann problem

$$(\eta - \Delta)u = f \quad \partial_n u\big|_{\theta=\theta_0} \equiv 0 \quad \partial_n u\big|_{\theta=\theta_+} \equiv 0\,. \tag{37}$$

Assume again that the consistency is well satisfied around $r = 0$. Namely, by setting $e = v - u$,

$$(\eta - \Delta)e = \rho \quad \partial_n e\big|_{\theta=\theta_0} \equiv 0 \quad \partial_n e\big|_{\theta=\theta_+} \equiv 0\,,$$

[5]The coefficients c_1, c_2 and the exponents of the parameter h are obtained by studying the asymptotics of the optimal interface conditions in the regular case when the uniform mesh size h goes to 0. Those results are detailed in [4].

with $\rho(r,\theta) = O(r^\infty)$. Then Proposition 5.2 says that even in this case

$$e(r,\theta) = a_0 + a_1 r^{\frac{1}{x_0}} \cos\left(\frac{\theta - \theta_0}{x_0}\right) + o(r^{\frac{1}{x_0}}) \tag{38}$$

with $x_0 = \frac{\theta_+ - \theta_0}{\pi}$, and $a_0, a_1 \in \mathbb{R}$. According to the definition 5.4, the asymptotic type in Ω for $\gamma < \frac{2}{x_0}$ is $\mathcal{T} = \Big((0,1,1); (\frac{i}{x_0}, 1, \cos(\frac{\theta-\theta_0}{x_0}))\Big)$. It describes the first term in the asymptotic expansion of the solution u to (37) with a vanishing right-hand side f.

b) Subproblem in Ω_1.

We focus on the subdomain Ω_1. The treatment of Ω_2 is similar. Like for the Dirichlet problem, we are led to consider the homogeneous problem:

$$\begin{cases} \Big(\partial_\theta^2 + (r\partial_r)^2\Big) e_1^{n+1}(r,\theta) & = 0 \\ \partial_\theta e_1^{n+1}(r,\theta_+) & = 0 \\ \Big(\partial_\theta + \frac{\alpha_-}{2}(r\partial_r)^2\Big) e_1^{n+1}(r,\theta_-) & = g_-(r) \end{cases} \tag{39}$$

where $g_-(r) = \Big(\partial_\theta + \frac{\alpha_-}{2}(r\partial_r)^2\Big) e_2^n(r,\theta_-)$.

After taking the Mellin transform, we get the z-dependent problem derived from the principal part

$$(\partial_\theta^2 - z^2)\widehat{e}_1(z,\theta) = 0 \qquad \partial_\theta \widehat{e}_1(z,\theta_+) = 0 \qquad (\partial_\theta - \frac{\alpha_-}{2}z^2)\widehat{e}_1(z,\theta_-) = \widehat{g}_-(z)\,, \tag{40}$$

whose solution is

$$\begin{aligned} & a(z)e^{z(\theta-\theta_-)} + b(z)e^{-z(\theta-\theta_+)} \\ & a(z) = \mathcal{R}(z)\widehat{g}_-(z),\ b(z) = a(z)e^{z(\theta_+-\theta_-)}, \\ \text{with} \quad & \mathcal{R}(z) = \frac{1}{z}\Big[(1 - \frac{\alpha_-}{2}z) - (1 + \frac{\alpha_-}{2}z)e^{2z(\theta_+-\theta_-)}\Big]^{-1}. \end{aligned}$$

We recall that the solution $u \in H^1$ implies that the multiplicity of the pole of Mu at $z = 0$ is 1 and permits to focus on the poles with positive imaginary part.

Proposition 7.3. *The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are the purely imaginary complex numbers $z = it$, with $t > 0$ and*

$$\tan(\pi x t) = -\frac{\alpha_-}{2}t\,, \quad x = \frac{\theta_+ - \theta_-}{\pi}, \tag{41}$$

whose positive solutions are denoted by $t_k, k \in \mathbb{N}^*$, in the increasing order.

Proof. In order to determine the poles of $\mathcal{R}(z)$, it is enough to solve the equation

$$(1-\frac{\alpha_-}{2}z)-(1+\frac{\alpha_-}{2}z)e^{2z(\theta_+-\theta_-)}=0\,.$$

Therefore the poles are solutions of

$$e^{2z(\theta_+-\theta_-)}=\frac{1-\frac{\alpha_-}{2}z}{1+\frac{\alpha_-}{2}z}\,. \tag{42}$$

As $z\in\mathbb{C}$, we note $z=r+it$ with $r,t\in\mathbb{R}$. Taking the module of (42) we obtain

$$e^{4r(\theta_+-\theta_-)}=\varphi_{\alpha_-}(r) \tag{43}$$

where

$$\varphi_{\alpha_-}(r)=\frac{(1-\frac{\alpha_-}{2}r)^2+(\frac{\alpha_-}{2})^2t^2}{(1+\frac{\alpha_-}{2}r)^2+(\frac{\alpha_-}{2})^2t^2}\,.$$

Since $\theta_+>\theta_-$ then we can deduce:

- If $r>0$ we get

  $$e^{4r(\theta_+-\theta_-)}>1 \text{ and } \varphi_{\alpha_-}(r)<1$$

  therefore one cannot have $r>0\,.$

- If $r<0$ we get

  $$e^{4r(\theta_+-\theta_-)}<1 \text{ and } \varphi_{\alpha_-}(r)>1$$

  therefore we cannot have $r<0\,.$

Therefore the equality (43) implies $r=0\,.$ The poles with a positive imaginary part have the form $z=it$, with $t>0$ and

$$e^{2it(\theta_+-\theta_-)}=\frac{1-i\frac{\alpha_-}{2}t}{1+i\frac{\alpha_-}{2}t}. \tag{44}$$

Writing the second member of (44) in the exponential form

$$e^{2it(\theta_+-\theta_-)}=e^{-2i\arctan(\frac{\alpha_-}{2}t)} \tag{45}$$

Hence t satisfies (41). ◻

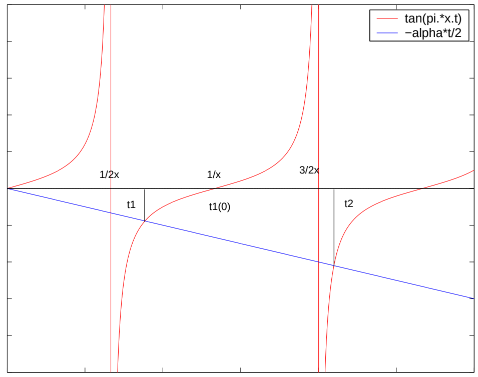

Figure 4: The first poles for the Neumann problem.

The best way to study the 2 first positive solutions of (41) is the graphical representation of $t \to \tan(\pi x t)$ and $t \to -\frac{\alpha_-}{2}t$ (see Figure 4). Below is the discussion between the two possible treatments of the first pole it_1 according to the general strategy presented in Section 6.

1. If the first approach which consists in canceling the first pole (it_1) is possible, the first active artificial pole becomes it_2 with $t_2 > \frac{3}{2x}$.

2. The second approach consists in pushing as far as possible the first pole it_1 from $z = 0$. In practice $\alpha_- > 0$ is taken very small so that $t_1 \simeq \frac{1}{x}$. Asymptotically this coincides with the choice of Neumann interface boundary conditions in the vicinity of the corner and the matching (7)(8) has to be modified.

When it is possible the first approach with $t_2 > \frac{3}{2x}$ is always better than the second one with $t_1 < \frac{1}{x}$.

c) Optimization of α_-.

In the subdomain Ω_1 and for a general right-hand side in (39), the first artificial term in the asymptotic expansion of e_1^{n+1} appears with the factor r^{t_1}, with t_1 defined in Proposition 7.3. The first (and most efficient) approach assumes that at step n the error as the natural asymptotic type associated with the global problem:

$$e_2^n(r,\theta) = a_0 + a_1 r^{\frac{1}{x_0}} \cos(\frac{\theta-\theta_0}{x_0}) + o(r^{\frac{1}{x_0}}).$$

With an additional truncation $\{r \leq R\}$, we set like in the domain decomposition algorithm

$$g_-(r) = 1_{\{r\leq R\}}\Big(\partial_\theta + \frac{\alpha_-}{2}(r\partial_r)^2\Big)e_2^n(r,\theta_-).$$

After forgetting the $o(r^{\frac{1}{x_0}})$ remainder and taking the Mellin transform, this provides

$$\widehat{g}_-(z) = a_1 \frac{R^{\frac{1}{x_0}+iz}}{\frac{1}{x_0}+iz} \Big[-\frac{1}{x_0}\sin(\frac{\theta_- - \theta_0}{x_0}) + \frac{\alpha_-}{2x_0^2}\cos(\frac{\theta_- - \theta_0}{x_0}) \Big] . \tag{46}$$

Here the cancellation of the first pole it_1 is reduced to the simple condition

$$\widehat{g}_-(it_1) = 0 . \tag{47}$$

Proposition 7.4. *The equation (47) is satisfied if and only if the parameter $\alpha_- \in \mathbb{R}_+^*$ satisfy*

$$\alpha_- = -2x_0 \tan(\frac{\pi x}{x_0}), \quad \frac{x_0}{2} < x \leq x_0 . \tag{48}$$

Proof. The equation (47) does not depend on the truncation parameter R and reads simply

$$-\frac{1}{x_0}\sin(\frac{\theta_- - \theta_0}{x_0}) + \frac{\alpha_-}{2x_0^2}\cos(\frac{\theta_- - \theta_0}{x_0}) = 0 .$$

This yields the result. □

We conclude this paragraph with some remarks about the implementation in the domain decomposition :

1. The solution (48) to the equation (47) can be non negative only when $\frac{x_0}{2} < x \leq x_0$. When the full domain Ω is split into two complementary sectorial domains Ω_1 and Ω_2, the condition has to be tested for $x = \frac{\theta_+ - \theta_-}{\pi}$ and $x_0 - x = \frac{\theta_- - \theta_0}{\pi}$. The answer can be affirmative for only one of them by excluding the symmetric decomposition.

2. By assuming in Ω_1, $x < \frac{x_0}{2}$, the first approach in Ω_1 cannot be applied. With the second one, i.e. by pushing as far as possible the first pole it_1 from $z = 0$, we find $\alpha_- = 0$ and $t_1 = \frac{1}{x(\Omega_1)}$.
   In practice it is enough to take α_- small. Graphically, since the graph of $\tan(\pi x t)$ are located below its tangent at the point $(\frac{1}{x}, 0)$ for $t \in (\frac{1}{2x}, \frac{1}{x})$, it is seen that the first pole t_1 (in fact it_1) appears for $t_1 > T_1$ with

$$\begin{aligned} & \pi x(T_1 - \frac{1}{x}) = -\frac{\alpha_-}{2} T_1 \\ \text{or} \qquad & t_1 > T_1 > \frac{2\pi}{\pi x_0 + \alpha_-} . \end{aligned}$$

   Hence taking $\alpha_- < 2\pi - \pi x_0 = 2\pi - (\theta_+ - \theta_0)$, implies $t_1 > T_1 > 1$, which is good enough.

3. The assumption $\frac{x}{x_0} < 1/2$ implies $1 \geq \frac{x_0 - x}{x_0} > 1/2$. Thus the first approach can be used in the second subdomain Ω_2.

4. For the domain decomposition into two non symmetric domains, this combination of the two approaches permits to expect after enough iterations a matching of the domain decomposition approximation and the full solution up to $O(r^{\min\{t_2(\Omega_2), t_1(\Omega_1)\}})$.

7.2 Improved interface conditions for interior artificial corner.

The decomposition of a regular domain into non overlapping subdomains, can produce interior artificial corners on the interfaces. This makes appear artificial singularities present in the solution of the auxiliary limits problems. Meanwhile the interior regularity of elliptic problems says here that $u \in H^{m+2}(\Omega)$ when $f \in H^m(\Omega)$.

We also noticed in Section 3 that in this case the choice of the coefficients has to be done with the condition $\beta_\pm = 0$.

Here the full domain $\Omega = \mathbb{R}^2 = \mathbb{R}_+^* \times S^1 \cup \{O\}$ will be decomposed into two nonoverlapping sectorial subdomains $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ or three (or more) subdomains Ω_1, Ω_2 and Ω_3, $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$.

We take again $\Omega_1 = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ with $\theta_+ - \theta_- \in]0, 2\pi[$. With two domains, we have $\Omega_2 = \mathbb{R}_+^* \times (\theta_0, \theta_-)$ with $\theta_0 = \theta_+ - 2\pi$.

a) Asymptotic type in Ω.

Consider the full domain problem

$$(\eta - \Delta)u = f \text{ in } \Omega = \mathbb{R}^2 . \tag{49}$$

Assume again that the consistency is well satisfied around $r = 0$. Namely, by setting $e = v - u$,

$$(\eta - \Delta)e = \rho \text{ in } \Omega = \mathbb{R}^2 ,$$

with $\rho(r, \theta) = O(r^\infty)$. Kondratiev theory (see Proposition 5.2) which coincides here with the interior regularity analysis for elliptic equations says that the asymptotic type of the solution is given by some Taylor expansion:

$$e(r, \theta) = b_0 + r\Big[b_1 \cos(\theta) + b_2 \sin(\theta)\Big] + o(r), \quad b_i, i = 0 \ldots 2 \in \mathbb{R} . \tag{50}$$

According Definition 5.4 which contains the Taylor expansion of a function of class $\mathcal{C}^\infty$ in an interior point in open set. It is enough to replace the interval (θ_-, θ_+) by the circle S^1. The asymptotic type in Ω is $\mathcal{T} = \Big((0,1,1);(i,2,(\cos(\theta),\sin(\theta)))\Big)$.

b) Subproblem in Ω_1.

Again, we first focus on one subdomain Ω_1, the treatment of the others ones Ω_2 (Ω_3,...)being similar. In the two subdomains case, the error $e_1^{n+1} = u_1^{n+1} - u$ solves a boundary value problem with the selected interface conditions:

$$\begin{cases} \Big(\eta - \frac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2)\Big)e_1^{n+1}(r,\theta) &= 0 \\ \Big(\frac{1}{r}\partial_\theta + \frac{1}{2}\partial_r(\tilde{\alpha}_-(r)\partial_r)\Big)e_1^{n+1}(r,\theta_-) &= g_-(r) \\ \Big(\frac{1}{r}\partial_\theta - \frac{1}{2}\partial_r(\tilde{\alpha}_+(r)\partial_r)\Big)e_1^{n+1}(r,\theta_+) &= g_+(r) \end{cases} \tag{51}$$

where

$$\begin{aligned} g_-(r) &= \Big(\tfrac{1}{r}\partial_\theta + \tfrac{1}{2}\partial_r(\tilde{\alpha}_-(r)\partial_r)\Big)e_2^n(r,\theta_-)\,, \\ g_+(r) &= \Big(\tfrac{1}{r}\partial_\theta - \tfrac{1}{2}\partial_r(\tilde{\alpha}_+(r)\partial_r)\Big)e_2^n(r,\theta_+)\,. \end{aligned}$$

We work here with

$$\alpha_+ = \alpha_- = \alpha_1\,, \quad (\beta_\pm = 0)\,,$$

and we will show later that this hypothesis is not restrictive for the optimization of (α_-, α_+). Following Kondratiev (see Section 5), we are led to consider the z-dependent problem derived from the principal part

$$(\partial_\theta^2 - z^2)\widehat{e}_1(z,\theta) = 0, \quad (\partial_\theta - \frac{\alpha_1}{2}z^2)\widehat{e}_1(z,\theta_-) = \widehat{g}_-(z), \quad (\partial_\theta + \frac{\alpha_1}{2}z^2)\widehat{e}_1(z,\theta_+) = \widehat{g}_+(z)\,. \tag{52}$$

whose solution is

$$\begin{aligned} &a(z)e^{z(\theta-\theta_-)} + b(z)e^{-z(\theta-\theta_+)} \\ &\begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = \frac{1}{z}\mathcal{R}(z)\begin{pmatrix} \widehat{g}_+(z) \\ \widehat{g}_-(z) \end{pmatrix} \\ \text{with} \qquad &\mathcal{R}(z) = \begin{pmatrix} (1+\frac{\alpha_1}{2}z)e^{z(\theta_+-\theta_-)} & -1+\frac{\alpha_1}{2}z \\ 1-\frac{\alpha_1}{2}z & -(1+\frac{\alpha_1}{2}z)e^{z(\theta_+-\theta_-)} \end{pmatrix}^{-1}. \end{aligned}$$

Proposition 7.5. *The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are the purely imaginary complex numbers $z = it$, with $t > 0$ and*

$$\tan\left(\frac{\pi x t}{2}\right) = \frac{2}{\alpha_1 t} \tag{53}$$

$$\text{or} \qquad \tan\left(\frac{\pi x t}{2}\right) = -\frac{\alpha_1}{2} t \tag{54}$$

with $x = \frac{\theta_+ - \theta_-}{\pi}$, whose positive solutions are denoted by $t_k, k \in \mathbb{N}^*$, in the increasing order.

Proof. The poles are solutions of

$$\det\begin{pmatrix} (1+\frac{\alpha_1}{2}z)e^{z(\theta_+-\theta_-)} & -1+\frac{\alpha_1}{2}z \\ 1-\frac{\alpha_1}{2}z & -(1+\frac{\alpha_1}{2}z)e^{z(\theta_+-\theta_-)} \end{pmatrix} = 0\,.$$

An obvious computation gives

$$e^{2z(\theta_+-\theta_+)} = \frac{(1-\frac{\alpha_1}{2}z)^2}{(1+\frac{\alpha_1}{2}z)^2}\,. \tag{55}$$

As $z \in \mathbb{C}$, we note $z = r + it$ with $r, t \in \mathbb{R}$. Taking the module of (55) we obtain

$$e^{2r(\theta_+-\theta_-)} = \varphi_{\alpha_1}(r) \tag{56}$$

where

$$\varphi_{\alpha_1} = \frac{(1-\frac{\alpha_1}{2}r)^2 + (\frac{\alpha_1}{2})^2 t^2}{(1+\frac{\alpha_1}{2}r)^2 + (\frac{\alpha_1}{2})^2 t^2}\,.$$

According to the proof of Proposition 7.3, $z = it$ with $t \in \mathbb{R}$. Hence t satisfy

$$e^{2it(\theta_+-\theta_+)} = \left(\frac{1-\frac{\alpha_1}{2}it}{1+\frac{\alpha_1}{2}it}\right)^2. \tag{57}$$

Then by setting

$$\lambda(z) = \frac{1-\frac{\alpha_1}{2}z}{1+\frac{\alpha_1}{2}z}e^{-z(\theta_+-\theta_-)}, \tag{58}$$

$z = it$ is a pole if and only if $\lambda(it) = \pm 1$. Thus we look further at the two equations corresponding to $\lambda(it) = +1$ or $\lambda(it) = -1, t > 0$. The same method as for Proposition 7.3 gives:

1. $\lambda(it) = 1$. Then the equation (57) becomes $\tan\left(\frac{t(\theta_+-\theta_-)}{2}\right) = -\frac{\alpha_1 t}{2}$.

2. $\lambda(it) = -1$. Then the equation (57) becomes $\tan\left(\frac{t(\theta_+-\theta_-)}{2}\right) = \frac{2}{\alpha_1 t}$.

□

The question which arises: Which equations from (53),(54) satisfy the first pole ?. Again the best way to understand which equation provides the first solution $t_1 > 0$ is the graphical representation of $t \longrightarrow \tan(\frac{\pi x t}{2})$, $t \longrightarrow \frac{2}{\alpha_1 t}$, and $t \longrightarrow -\frac{\alpha_1}{2} t$, with $x = \frac{\theta_+ - \theta_-}{\pi}$.

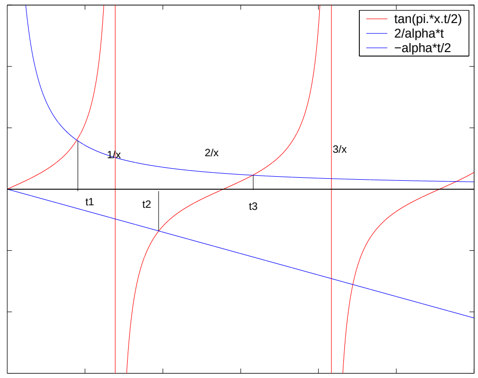

Figure 5: Case of $\alpha_1 > 0$

According to Figure 5, the first pole it_1 is associated with the equation

$$\tan\left(\frac{\pi x t}{2}\right) = \frac{2}{\alpha_1 t}, \quad (\lambda(it_1 = -1))\,.$$

c) Optimization of $\alpha_+ = \alpha_- = \alpha_1$.

In the subdomain Ω_1 and for a general right-hand side in (51), the first artificial term in the asymptotic expansion of e_1^{n+1} appears with the factor r^{t_1}, with t_1 defined in Proposition 7.5 . The first (and most efficient) approach assumes that at step n the error as the natural asymptotic type associated with the global problem:

$$e_2^n(r,\theta) = b_0 + r\big[b_1\cos(\theta) + b_2\sin(\theta)\big] + o(r)\,.$$

With an additional truncation in $\{r \leq R\}$, we set like in the domain decomposition algorithm

$$\begin{aligned} g_+(r) \quad &= 1_{\{r\leq R\}}\Big(\partial_\theta - \tfrac{\alpha_1}{2}(r\partial_r)^2\Big)e_2^n(r,\theta_+) \\ g_-(r) \quad &= 1_{\{r\leq R\}}\Big(\partial_\theta + \tfrac{\alpha_1}{2}(r\partial_r)^2\Big)e_2^n(r,\theta_-) \end{aligned}$$

where $R \in \mathbb{R}_+^*$. The general strategy of matching the asymptotic types is translated here into: if g_+ and g_- are associated with a regular solution outside Ω_1 then the solution to (51) in Ω_1 must have a good regularity.

After working with the first order Taylor expansion, forgetting the $o(r)$ remainder and taking the Mellin transform, this provides

$$\begin{aligned}\widehat{g}_+(z) &= \tfrac{R^{1+iz}}{1+iz}\Big[b_1(-\sin\theta_+ - \tfrac{\alpha_1}{2}\cos\theta_+) + b_2(\cos\theta_+ - \tfrac{\alpha_1}{2}\sin\theta_+)\Big];\\ \widehat{g}_-(z) &= \tfrac{R^{1+iz}}{1+iz}\Big[b_1(-\sin\theta_- + \tfrac{\alpha_1}{2}\cos\theta_-) + b_2(\cos\theta_- + \tfrac{\alpha_1}{2}\sin\theta_-)\Big].\end{aligned}$$

We note that the boundary conditions do not contains a constant term, therefore $g_\pm$ do not depend any more a value into zero of e_2^n.

Our goal is to see whether there exists α_1 such as

$$\mathcal{R}(z)\begin{pmatrix}\widehat{g}_+(z)\\ \widehat{g}_-(z)\end{pmatrix} \text{ does not have any more pole on } it_1, (t_1>0).$$

Here the cancellation of the first artificial pole it_1 is reduced to the simple condition

$$\widehat{g}_+(it_1) = \lambda(it_1)\widehat{g}_-(it_1), \quad \lambda(it_1) = -1\,. \tag{59}$$

Proposition 7.6. *The equation (59) gives*

$$\alpha_1 = \frac{2}{\tan(\frac{\pi x}{2})}, \ \textit{for } x \in]0,1[\tag{60}$$

with $x = \frac{\theta_+ - \theta_-}{\pi}$.

Proof. The equation (59) does not depend on the truncation parameter R and reads simply

$$\begin{cases}\dfrac{\alpha_1}{2}(\cos\theta_- - \cos\theta_+) = \sin\theta_+ + \sin\theta_-\\[2ex] \dfrac{\alpha_1}{2}(\sin\theta_+ - \sin\theta_-) = \cos\theta_+ + \cos\theta_-\end{cases} \tag{61}$$

The two equations of (61) are equivalent and this yields the result. □

While working with only one parameter in the interface conditions, the system (61) has no positive solution, $\alpha_1 > 0$, when the sector is not convex, $\theta_+ - \theta_- > \pi$. In the next paragraph, we will show that even with two parameters we cannot cancel the first pole in a non-convex sector.

d) Study with two parameters.

One works in this part with $\alpha_\pm$ in the interface conditions. According to the study already made one has

$$(\partial_\theta^2 - z^2)\widehat{e}_1(z,\theta) = 0, \quad (\partial_\theta - \frac{\alpha_-}{2}z^2)\widehat{e}_1(z,\theta_-) = \widehat{g}_-(z), \quad (\partial_\theta + \frac{\alpha_+}{2}z^2)\widehat{e}_1(z,\theta_+) = \widehat{g}_+(z) \tag{62}$$

whose solution is

$$\begin{aligned} & a(z)e^{z(\theta-\theta_-)} + b(z)e^{-z(\theta-\theta_+)} \\ & \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = \frac{1}{z}\mathcal{R}(z)\begin{pmatrix} \widehat{g}_+(z) \\ \widehat{g}_-(z) \end{pmatrix} \\ \text{with} \quad & \mathcal{R}(z) = \begin{pmatrix} (1+\frac{\alpha_+}{2}z)e^{z(\theta_+-\theta_-)} & -1+\frac{\alpha_+}{2}z \\ 1-\frac{\alpha_-}{2}z & -(1+\frac{\alpha_-}{2}z)e^{z(\theta_+-\theta_-)} \end{pmatrix}^{-1}. \end{aligned}$$

Proposition 7.7. *The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are the purely imaginary complex numbers $z = it$, with*

$$\left\{ \begin{array}{ll} \tan(\pi x t) & = \frac{\frac{\alpha_+ + \alpha_-}{2}t}{\frac{\alpha_+\alpha_-}{4}t^2 - 1}, t > 0 \\ \quad or & \\ \quad \pi x t & \equiv \frac{\pi}{2}(\pi) \ if \ \frac{\alpha_+\alpha_-}{4}t^2 - 1 = 0 \end{array} \right. \tag{63}$$

with $x = \frac{\theta_+ - \theta_-}{\pi}$, whose positive solutions are denoted by $t_k, k \in \mathbb{N}^*$, in the increasing order.

Proof. The same argument as the proposition 7.5 give that the poles are solution of

$$e^{2z(\theta_+-\theta_-)} = \frac{(1-\frac{\alpha_+}{2}z)}{(1+\frac{\alpha_+}{2}z)}\frac{(1-\frac{\alpha_-}{2}z)}{(1+\frac{\alpha_-}{2}z)}. \tag{64}$$

If we replace z with $r+it$ this lead to

$$e^{4r(\theta_+-\theta_-)} = \varphi_{\alpha_+}(r)\,\varphi_{\alpha_-}(r)\,. \tag{65}$$

We recall that

$$\varphi_{\alpha_\pm}(r) = \frac{(1-\frac{\alpha_\pm}{2}r)^2 + (\frac{\alpha_\pm}{2})^2t^2}{(1+\frac{\alpha_\pm}{2}r)^2 + (\frac{\alpha_\pm}{2})^2t^2}\,.$$

It is noticed that for $r > 0$, we have $\varphi_{\alpha_\pm}(r) < 1$ but $e^{4r(\theta_+-\theta_-)} > 1$, thus we cannot have $r > 0$, the same argument gives that r cannot be negative.

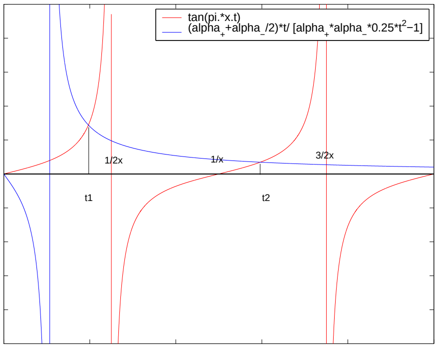

Figure 6: Case of $\frac{2}{\sqrt{\alpha_-\alpha_+}} < \frac{1}{2x}$.

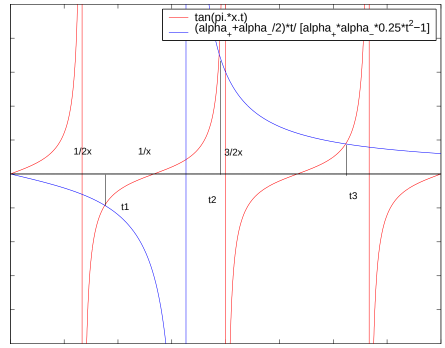

Figure 7: Case of $\frac{2}{\sqrt{\alpha_-\alpha_+}} > \frac{1}{2x}$.

Therefore the equality (65) implies $r = 0$.

The poles with a positive imaginary part have the form $z = it$, with $t > 0$ and

$$e^{2it(\theta_+-\theta_-)} = \frac{(1 - \frac{\alpha_+}{2}it)(1 - \frac{\alpha_-}{2}it)}{(1 + \frac{\alpha_+}{2}it)(1 + \frac{\alpha_-}{2}it)}. \tag{66}$$

Writing the second member of the equality (66) in the exponential form then by equality of arguments we deduce that t satisfy (63). $\square$

The best way to study the two first positive solutions of (63) is the graphical representation of $t \to \tan(\pi x t)$ and $t \to \frac{\frac{\alpha_+ + \alpha_-}{2} t}{\frac{\alpha_+\alpha_-}{4} t^2 - 1}$. There are 2 different cases according to $\frac{2}{\sqrt{\alpha_-\alpha_+}} < \frac{1}{2x}$ or $\frac{2}{\sqrt{\alpha_-\alpha_+}} > \frac{1}{2x}$ which compares the position of the first vertical asymptote (see Figures 6 and 7).

Below is the discussion between the two possible treatments of the first pole it_1 according to the general strategy presented in Section 6.

1. If $\frac{2}{\sqrt{\alpha_+\alpha_-}} < \frac{1}{2x}$, the poles check

$$k = 1,\ t_1 \in \ ]\tfrac{2}{\sqrt{\alpha_+\alpha_-}}, \tfrac{1}{2x}[$$

$$k \geq 2,\ t_k \in \ ]\tfrac{k-1}{x}, \tfrac{k-\frac{1}{2}}{x}[$$

2. If $\frac{2}{\sqrt{\alpha_+\alpha_-}} > \frac{1}{2x}$ the poles check

$$k = 1,\ t_1 \in \ ]\tfrac{1}{2x}, \min(\tfrac{2}{\sqrt{\alpha_+\alpha_-}}, \tfrac{1}{x})[$$

$$k \geq 2,\ t_k \geq \ \tfrac{1}{x}$$

3. In the borderline cases $\frac{2}{\sqrt{\alpha_+\alpha_-}} = \frac{1}{2x}$ the first pole is given by $t_1 = \frac{1}{2x}$.

4. The second approach consists in pushing as far as possible the first pole it_1 from $z = 0$. In practice it is done by choosing $\alpha_+ > 0$ and $\alpha_- > 0$ very small so that $t_1 \simeq \frac{1}{x}$. Asymptotically this coincides with the choice of Neumann interface boundary conditions in the vicinity of the corner and the matching (7)(8) has to be modified. When it is possible the first approach with $t_2 > \frac{1}{x}$ is always better than the second one with $t_1 < \frac{1}{x}$.

The same computations as in the former case $\alpha_+ = \alpha_- = \alpha_1$ give

$$\begin{aligned}
\widehat{g}_+(z) &= \tfrac{R^{1+iz}}{1+iz}\Big[b_1(-\sin\theta_+ - \tfrac{\alpha_+}{2}\cos\theta_+) + b_2(\cos\theta_+ - \tfrac{\alpha_+}{2}\sin\theta_+)\Big]\,,\\
\widehat{g}_-(z) &= \tfrac{R^{1+iz}}{1+iz}\Big[b_1(-\sin\theta_- + \tfrac{\alpha_-}{2}\cos\theta_-) + b_2(\cos\theta_- + \tfrac{\alpha_-}{2}\sin\theta_-)\Big]\,.
\end{aligned}$$

The same procedure as Proposition 7.6 gives: the cancellation of the first artificial pole it_1 is reduced to the simple condition

$$\widehat{g}_+(it_1) = \lambda(it_1)\widehat{g}_-(it_1) \tag{67}$$

with

$$\lambda(it_1) = \frac{1 - i\frac{\alpha_+}{2}t_1}{1 + i\frac{\alpha_-}{2}t_1}e^{-it_1(\theta_+-\theta_-)} \in \mathbb{R}\,.$$

The Figures 6 and 7 show that in all the cases $t_1 < \frac{\pi}{\theta_+-\theta_-}$, therefore for a non-convex sector we have $t_1 < 1$.

Proposition 7.8. *Assume that $\theta_+ - \theta_- > \pi$. If $\widehat{g}_+(it_1) = \lambda(it_1)\widehat{g}_-(it_1)$ then $\alpha_+ = \alpha_-$.*

Proof. The equation (67) does not depend on the truncation parameter R and reads simply

$$\left\{\begin{aligned}
\lambda(it_1)\Big(\sin\theta_- - \tfrac{\alpha_-}{2}\cos\theta_-\Big) &= \Big(\tfrac{\alpha_+}{2}\cos\theta_+ + \sin\theta_+\Big)\\
\lambda(it_1)\Big(\cos\theta_- + \tfrac{\alpha_-}{2}\sin\theta_-\Big) &= -\Big(\tfrac{\alpha_+}{2}\sin\theta_+ - \cos\theta_+\Big)
\end{aligned}\right. \tag{68}$$

this lead to consider

$$\left\{\begin{aligned}
|\lambda(it_1)|^2\Big(\sin\theta_- - \tfrac{\alpha_-}{2}\cos\theta_-\Big)^2 &= \Big(\tfrac{\alpha_+}{2}\cos\theta_+ + \sin\theta_+\Big)^2\\
|\lambda(it_1)|^2\Big(\cos\theta_- + \tfrac{\alpha_-}{2}\sin\theta_-\Big)^2 &= \Big(\tfrac{\alpha_+}{2}\sin\theta_+ - \cos\theta_+\Big)^2
\end{aligned}\right. \tag{69}$$

with

$$|\lambda(it_1)|^2 = \frac{1+(\frac{\alpha_+}{2})^2 t_1^2}{1+(\frac{\alpha_-}{2})^2 t_1^2}\,.$$

This lead to

$$|\lambda(it_1)|^2\left[(\sin\theta_- - \frac{\alpha_-}{2}\cos\theta_-)^2+(\cos\theta_- + \frac{\alpha_-}{2}\sin\theta_-)^2\right] = (\frac{\alpha_+}{2}\cos\theta_+ + \sin\theta_+)^2+(\frac{\alpha_+}{2}\sin\theta_+ - \cos\theta_+)^2 \tag{70}$$

with

$$|\lambda(it_1)|^2 = \frac{1+(\frac{\alpha_+}{2})^2 t_1^2}{1+(\frac{\alpha_-}{2})^2 t_1^2}\,.$$

The equation (70) gives

$$\left[(\frac{\alpha_+}{2})^2 - (\frac{\alpha_-}{2})^2\right](t_1^2-1) = 0. \tag{71}$$

This yields the result. □

We end this general presentation by two remarks.

1. For a decomposition into two subdomains, with Ω_1 convex and Ω_2 non-convex. In Ω_1 we can choose $\alpha_1 > 0$ and its value is given by (60). In the second one we use the second approach which consists in pushing as far as possible the first pole $it_1(\Omega_2)$ from $z = 0$. A good choice is α_1 very small and $t_1(\Omega_2) \simeq \frac{1}{x(\Omega_2)}$. This combination of the two approaches permits to expect after enough iterations a matching of the domain decomposition approximation and the full solution up to $O(r^{\min\{t_2(\Omega_1),t_1(\Omega_2)\}})$.

2. Another way, is to make a decomposition in three convex subdomains. In such cases, the first approach permits to expect after enough iterations a matching of the domain decomposition approximation and one hopes that the full solution up to $O(r^{\min\{t_2(\Omega_1),t_2(\Omega_2),t_2(\Omega_3)\}})$. The notation $t_2(\Omega_j)$ refers to the second positive solution to (60) adapted to the domain $\Omega_j, j = 1,\ldots,3$.

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